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On the space energy distribution of slowed-down neutrons. II

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With 2 figures in the text

1. Introduction

The purpose of this paper is a calculation of the stationary space energy distribution of slow neutrons in a moderator at all distances from the source.

Exact methods of calculating the density of slowed-down neutrons have been made available by the derivation of the general equation governing the diffusion and slowing-down of neutrons[1], i.e. the transport equation, and its solution for the case of an isotropic point source in an infinite medium by means of a Fourier-Laplace transformation[1]. The elementary age-theory[2] gives a general formula for the slow neutron density, which is assumed to be valid at small distances r from the source, for a slowly varying mean free path $\lambda(E)$ of the neutrons (E = neutron energy), and best for large values of M (= mass of the scattering atomic nuclei measured in units of the neutron mass). In other cases, however, and always for large r, this formula may give considerable errors. Therefore it is important to make a calculation of the neutron density which is based on the general theory, and for this purpose the Fourier-Laplace transform of the transport equation seems to be the most convenient starting-point.

So far, most calculations of the distribution of slowed-down neutrons outside the range of validity of age-theory have been made assuming $\lambda(E) = \lambda_0 =$ = constant. Important results for this case have been obtained by WALLER [13], WICK [15] and others. The problem was also treated by the present author in a previous paper[3], of which this paper is a sequel. The papers quoted give a fairly complete picture of the slow neutron density in the constant λ case, including analytic expressions for this function at all distances r.

When the variation of $\lambda(E)$ must be taken into account, the difficulty of calculating the distribution function increases considerably in the general case. A lot of work has therefore been done instead on calculations of spatial moments of the distribution function [8], which is easier than the computation of the function itself. Although this gives some idea of the space distribution, an explicit expression of course would give much better information. But there exist only a few results beyond the age-theory formula for the distribution function. VERDE and WICK [14] have made such a calculation, but only for a very particular form of the function $\lambda(E)$. For the case when $\lambda(E)$ decreases with decreasing E, WICK [15] has obtained some important results and derived the asymptotic behaviour of the neutron density for $r \to \infty$.

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[1] See WALLER [12], [13] or MARSHAK [8].

[2] See MARSHAK [8], for instance.

[3] In the following this paper will be referred to as I, and formulas in it for example as I (1).

It is the purpose of this paper to extend the knowledge of the slow neutron density in the case of a variable mean free path. As far as possible we intend to calculate analytic formulas for the distribution function, thus completing the results obtained for constant λ. The assumptions made are the same as in I, and it seems unnecessary to repeat them here. The starting-point as in I will be the Fourier-Laplace transform of the transport-equation, and developments and analytic continuation in the variables of this transform will also be used. Previous results for the case of variable λ mentioned above, in particular those by WICK [15] will also be helpful.

There is an essential additional difficulty connected with the use of the Fourier-Laplace transformation in the case of non-constant λ. In fact, this does not seem to be useful at all unless special assumptions are made on the function $\lambda(E)$. On the other hand, we have no close theoretical formulas for $\lambda(E)$ (except for hydrogen) so that in any case this function must be taken from the neutron cross section measurements [1], [4]. By taking for $\lambda(E)$ a polynomial of sufficiently high degree, for instance, any accuracy desired may be obtained. It is still better, implying no restriction in practice to make the assumption first used by MARSHAK [8]

$$\lambda(E) = \sum_{\nu=0} \lambda_\nu \left(\frac{E}{E_0}\right)^{d_\nu} = \sum_{\nu=0} \lambda_\nu e^{-d_\nu x}. \tag{1}$$

Here λ_ν and d_ν are arbitrary constants, but $d_\nu \geq 0$. (E_0 = energy of the neutron source.) Further we shall always take $d_0 = 0$. The number of terms in (1) thus depends on the accuracy wanted. The case of constant λ is obtained in (1) by taking only the first term, $\lambda = \lambda_0$ and is a good approximation for sufficiently low values of E_0 $(E \leq E_0)$. In many cases of practical interest a λ variation with only two terms, those in λ_0 and λ_1, is a good approximation and means a considerable improvement over the assumption $\lambda = \lambda_0$. However, no special assumption is made now on the number of terms in (1), although the actual numerical applications to carbon and water were performed with this simplified expression for $\lambda(E)$.

If (1) is used for $\lambda(E)$, the Fourier-Laplace transformation is applicable. The system of equations which results from the transport equation after the transformation and an expansion in spherical harmonics was first given by MARSHAK [8] and has also been used earlier [5] by the present author:

$$\begin{aligned}(2n+1)\,h_n(\eta)\,f_n(\omega,\eta) + \frac{i\,\omega}{\lambda_0}\sum_{\nu=0}\lambda_\nu\,[(n+1)\,f_{n+1}(\omega,\eta+d_\nu) +{}\\ + n\,f_{n-1}(\omega,\eta+d_\nu)] = q(\omega,\eta)\,\delta_{n0}, \qquad (n = 0, 1, 2, \ldots).\end{aligned} \tag{2}$$

The form chosen here for these equations assumes for the mean free path of the slow neutrons λ_0 that $\lambda_0 \neq 0$, which is always fulfilled in practice. $h_n(\eta)$

are the functions given by I (4), (5) and (6). With the special assumptions made about the neutron source $q(\omega, \eta)$ is given by I (3). From (2) the functions $f_0(\omega, \eta), f_1(\omega, \eta), \ldots$ have to be calculated. Of these functions only f_0 is needed if we are merely interested in the total number of neutrons per unit volume, unit energy range $N_0(r, E)$. This is related to the collision density $F_0(r, E)$ which is ordinarily used, by an equation $N_0(r, E) = \lambda(E)\, v^{-1} F_0(r, E)$ (absorption is neglected, v = neutron velocity). The other functions $f_1, f_2, \ldots$ are only needed if we also want to study the distribution of the neutrons on the different directions of movement, but this will not be done here.

Having calculated $f_0(\omega, \eta)$, there remains for the computation of $F_0(r, E)$ the evaluation of the integral I (1), or

$$F_0(r, E) = -\frac{1}{2\pi^2 \lambda_0^2 r} \int\limits_{\eta_1 - i\infty}^{\eta_1 + i\infty} d\eta\, e^{x\eta} \int\limits_{-\infty}^{+\infty} d\omega\, \omega\, e^{i\omega R} f_0(\omega, \eta). \tag{3}$$

Here $x = \log E_0/E$ is the logarithmic energy variable and r is the distance from the source. For abbreviation we shall sometimes also use, as in I, the variable $R = r/\lambda_0$, i.e. here the distance measured with the slow neutron mean free path as the unit of length.

Before we start the calculations, we shall briefly sum up the main points in this paper. In Section 2 the physically important solution of (2) is determined. Here (4) is the essential equation. It implies that f_0 can be written as a product of a function g_0 and the continued fraction I (2) from the constant λ case. This makes it possible to use the same method as in I for one integration where the essential contribution to the integral comes from the same main pole and then apply the saddle-point method for the other integration. Section 2 then contains a detailed calculation of the function g_0 by means of a power series in ω^2. The crucial equations used in deriving g_0 are (5) and (6); it is also shown how g_0 may be given as a closed expression, eq. (7) ... (16). The rest of Section 2 is devoted to a discussion of g_0.

Section 3 contains a calculation of F_0 without special assumptions on $\lambda(E)$ (except (1)). It uses essentially the same methods of computation as in I, and gives an expression for F_0 where age-theory appears as the first approximation. Therefore it is useful only for sufficiently small distances and slowly varying $\lambda(E)$.

The main part of the paper is Section 4, where F_0 is calculated without the restrictions necessary for the expression given in Section 3. This is possible by assuming instead that $\lambda(E)$ is decreasing with decreasing E, which is the most important case in practice. Then the results by WICK [15] concerning the behaviour of f_0 can be used. They are first summarized, and then it is shown how they can be applied in combination with the general method of developments and analytic continuation in the variables, which is here, as in I, the main tool for computing the integrand of the saddle-point integral. Then F_0 is calculated, given by the main formula (37). This is completed for large distances by (48). Further we have for F_0 formula (44), valid within an intermediate domain, but not applicable for a too rapid variation of $\lambda(E)$. Finally the complete asymptotic formula for $r \to \infty$ is obtained (49). A discussion of F_0 concludes Section 4.

Section 5 is devoted to applications. First some generalities about the numerical calculations, performed only in a special case of (1), are given. Then numerical results for carbon and water are presented, including also a comparison with experiments for water.

2. Determination of the function $f_0(\omega, \eta)$

Solution of (2)

(2) is an infinite system of difference equations in two variables η and n. For $\lambda = \lambda_0$, i.e. $\lambda_1 = \lambda_2 = \cdots = 0$, η appears only as a parameter and we have only a system of difference equations in n. The solution of this system [12], [13] may be given as certain continued fractions as in I (2) for $f_0(\omega, \eta)$. In the following this function will be called $f_0^{(\lambda_0)}(\omega, \eta) = q/N_0(\omega, \eta)$ and in general $N_n(\omega, \eta)$ is the continued fraction $h_n(\eta) + \cdots$ contained in I (2). Now it is easily seen that the function $f_0(\omega, \eta)$ has the same poles (among others) as the function $f_0^{(\lambda_0)}(\omega, \eta)$. If we put

$$f_0(\omega, \eta) = f_0^{(\lambda_0)}(\omega, \eta)\, g_0(\omega, \eta), \tag{4}$$

this also appears from a method of calculating $f_0(\omega, \eta)$ directly in the form (4), where $g_0(\omega, \eta)$ is regular and $\neq 0$ in particular when $Re(\eta) \geq 1$ (the real part of η) and ω is near the main pole $\omega_0(\theta)$ $(\theta = \eta - 1)$ of $f_0^{(\lambda_0)}(\omega, \eta)$. This possible way of calculation is a generalization of a method used by the author in an earlier paper [5]. We may always obtain approximate solutions of (2) by breaking off the equations at a certain index k, and putting $f_n = 0$ for $n > k$, as the condition $\lim_{n \to \infty} f_n(\omega, \eta) = 0$ must be fulfilled for solutions of physical significance. The last equation $(n = k)$ then gives directly an approximate value of f_k, which contains as a factor the first approximation of the continued fraction $1/N_k$. Putting this value into the equation (2) for $n = k - 1$, we obtain a difference equation in η for the determination of f_{k-1} in terms of f_{k-2}. This equation is of a slightly more general form than the equation treated in the paper quoted ([5], eq. (15) and (16)) but may be solved using the same method. The result is the product of the second approximation of the continued fraction $1/N_{k-1}$ and a certain development, essentially a power series in ω^2, broken off for consistency with the term in ω^2. This value of f_{k-1} is then put into the equation for $n = k - 2$ in (2) and the same procedure is repeated until we finally get f_0 as the product of the $(k + 1)$th approximation of $f_0^{(\lambda_0)}$ and a development, determined up to terms in ω^{2k}. It is important to notice that this development contains the function h_0 only with the arguments $\eta + d_1$, $\eta + d_2$, $\eta + d_1 + d_2$, etc. This method of solution is always valid for sufficiently small values of $|\omega|$ and in passing to the limit $k \to \infty$ we obtain the form (4) for f_0, where g_0 is obtained as a power series in ω^2 which in particular does not contain $h_0(\eta)$ (but $h_0(\eta + d_1)$ etc.). This expression for f_0 is convenient for the evaluation of (3) because we have the same main pole $\omega = \omega_0(\theta)$ as in I and may perform one of the integrations by the method of residues, obtaining an integral similar to I (17). This contains a rapidly varying exponential function, making the application of the saddle-point method possible.

The method above for the solution of (2) thus proves the validity of the expression (4) for f_0 and some important properties of g_0. Unfortunately, the method is not convenient for an actual calculation of f_0 except for the first few approximations. It does not seem possible to obtain the general term in the power series of g_0 in this way. But instead we may try first to derive the development in ω^2 of all the function f_0 and then sum up the factor $f_0^{(\lambda_0)}$, thus obtaining g_0. Such a development of f_0 has, in fact, been used earlier [8] for the calculation of spatial moments of the neutron density, but has not, so far, seemed to be useful for the computation of the neutron density itself. However, combined with the transformation (4) and using analytic continuation, this development is very powerful, as will appear in this paper.

We now proceed to the solution of (2) with power series in ω^2. We try

$$(5) \qquad f_n(\omega, \eta) = (-1)^n \frac{n!}{(2n+1)!!} \sum_{k=n,\, n+2, \ldots} (i\omega)^k f_{nk}(\eta).$$

The factor in (5) is different from that used before [8], but with the choice made here the system of equations which follows from (2) and (5) has its simplest form. An easy computation shows that these equations are

$$(6) \quad f_{nk}(\eta) = \frac{1}{h_n(\eta)} \left[r_{n+1} \sum_{\nu=0} \frac{\lambda_\nu}{\lambda_0} f_{n+1,\, k-1}(\eta + d_\nu) + \sum_{\nu=0} \frac{\lambda_\nu}{\lambda_0} f_{n-1,\, k-1}(\eta + d_\nu) + q\, \delta_{n0}\, \delta_{k0} \right],$$

with

$$r_n = \frac{n^2}{4n^2 - 1}, \qquad (k = n,\, n+2,\, n+4, \ldots,\, n = 0, 1, 2, \ldots).$$

This is a recurrence relation for determination of $f_{nk}(\eta)$, starting with the value $f_{00}(\eta) = q/h_0(\eta)$. In particular, we may calculate as many coefficients in the development of $f_0(\omega, \eta)$, $f_{02}(\eta)$, $f_{04}(\eta)$, ... as we want, though with rapidly increasing difficulty.

Of course, it would be desirable if we could replace successive calculation by a closed expression for the solution of (6). By introduction of certain operators and convenient abbreviations for occurring sums, this is, in fact, also possible, and it seems worth while reproducing this solution here. For that purpose we first introduce an operator P_n, essentially performing a summation, by defining

$$(7) \qquad P_n \Phi(\eta) = \frac{1}{h_n(\eta)} \sum_{\nu=0} \frac{\lambda_\nu}{\lambda_0} \Phi(\eta + d_\nu),$$

where Φ is an arbitrary function of η. This abbreviation is immediately suggested by the system (6) which now takes the simple form

$$(8) \qquad f_{nk}(\eta) = r_{n+1} P_n f_{n+1,\, k-1}(\eta) + P_n f_{n-1,\, k-1}(\eta) + q\, \delta_{n0}\, \delta_{k0},$$
$$(k = n,\, n+2, \ldots,\, n = 0, 1, \ldots)$$

The solution of (8) will contain multiple summations, obtained by a repeated use of operators P_n on functions $1/h_m(\eta)$. Therefore it is convenient to introduce the following functions of η, depending further on an arbitrary number of indices $i, j, k, \ldots$

$$(9) \qquad D_{ijk\ldots}(\eta) = \frac{1}{h_i(\eta)} \sum_{\substack{n_j=0\\ n_k=0\\ \vdots}} \frac{\dfrac{\lambda_{n_j}}{\lambda_0} \cdot \dfrac{\lambda_{n_k}}{\lambda_0} \cdots}{h_j(\eta + d_{n_j})\, h_k(\eta + d_{n_j} + d_{n_k}) \cdots}.$$

We notice the simple relation, which follows at once from (7) and (9)

$$(10) \qquad P_n D_{ijk\ldots}(\eta) = D_{nijk\ldots}(\eta).$$

Now $f_{nk}(\eta)$ obviously is built up by sums of terms (9) multiplied with factors r_n, with different combinations of values of the indices $i, j, k \ldots$ The problem is to find the law according to which these combinations are made and then obtain an expression for the sum in a closed form. This can be accomplished by introducing operators Q_ν ($\nu = 1, 2, \ldots$), operating on the row of indices $i\,j\,k \ldots$ in the following way. For every operator Q_ν we insert in the row of indices a sign, say $\dot{;}_{\nu}$, which divides the row into two groups. All indices to the left of the sign $\dot{;}_{\nu}$ are left unchanged by Q_ν, which only operates on the indices to the right of the sign. These signs always occur in the row of indices in falling order ν from the left to the right, so that the sign $\dot{;}_1$ is furthest to the right. Now we define the effect of Q_ν: it performs a subtraction of 2 from the ν first indices to the right of the sign $\dot{;}_{\nu}$; further it transfers all signs $\dot{;}_{\nu}, \dot{;}_{\nu-1}, \ldots \dot{;}_1$ one step to the right, for example

$$Q_1 D_{i\dot{;}_1 jkl\ldots} = D_{ij-2\dot{;}_1 kl\ldots} \quad\text{and}\quad Q_2 D_{i\dot{;}_2 jk\dot{;}_1 lm\ldots} = D_{ij-2\dot{;}_2 k-2 l\dot{;}_1 m\ldots} \quad\text{etc.}$$

We may then define a product operator $Q_\nu \ldots Q_\varrho Q_\sigma$, which of course means a successive application of the operators $Q_\sigma, Q_\varrho, \ldots Q_\nu$, and in particular a power Q_ν^r ($r \geq 0$) is defined in this way ($Q_\nu^0 = 1$, i.e. it leaves the row of indices unchanged). Sums and differences of such products, also containing ordinary numbers as factors, are then immediately defined in the usual way.

Observing further that the operators P_n and Q_ν commute, and putting $k = n + 2\nu$, it is easily shown that the following expression gives the solution of (8)

$$(11) \qquad f_{n,n+2\nu}(\eta) = q \underset{\nu}{\Bigg[} \sum_{m_\nu=1}^{n+1} r_{m_\nu} \underset{\nu-1}{\Bigg[} \sum_{m_{\nu-1}=1}^{m_\nu+1} r_{m_{\nu-1}} \cdots \underset{2}{\Bigg[} \sum_{m_2=1}^{m_3+1} r_{m_2} \underset{1}{\Bigg[} \sum_{m_1=1}^{m_2+1} r_{m_1} Q_1^{m_2+1-m_1} \underset{1}{\Bigg]} \cdot$$

$$\cdot\, Q_2^{m_3+1-m_2} \underset{2}{\Bigg]} \cdots Q_{\nu-1}^{m_\nu+1-m_{\nu-1}} \underset{\nu-1}{\Bigg]} Q_\nu^{n+1-m_\nu} \underset{\nu}{\Bigg]} D(\eta)_{n\dot{;}_{\nu} n+1 \dot{;}_{\nu-1} \ldots n+\nu-1 \dot{;}_1 n+\nu\, n+\nu-1 \ldots 210},$$

$$(n, \nu = 0, 1, 2, \ldots).$$

In evaluating this expression we have to take the parenthesis ν with the summation in m_ν first, obtaining then a sum of $n+1$ terms of the same general type as (11), though according to the effect of $Q_\nu^{n+1-m_\nu}$ with different indices in D; only the term with $m_\nu = n + 1$ has the same indices as in (11). Then in each of these terms the summation in $m_{\nu-1}$ has to be performed. The procedure is repeated until the last operator Q_1 has been used and the summation in m_1 has been made. Now the term in (11) with $m_\nu = n + 1$ is in fact the result of applying the operator $r_{n+1} P_n$ on $f_{n+1, n+1+2(\nu-1)}(\eta)$ if (11) is used for this term, and the other terms with $m_\nu = 1 \ldots n$ are easily seen to be the result of P_n operating on $f_{n-1, n-1+2\nu}(\eta)$, when (11) is applied to this function. This proves that (11) is the solution of (8), since in addition the initial conditions for $n = \nu = 0$ are fulfilled by (11) which then gives $f_{00}(\eta) = q D_0(\eta) = q/h_0(\eta)$.

Here we are merely interested in (11) in the special case $n = 0$, and it was only a step on our way to obtain the form (4) for $f_0(\omega, \eta)$ and the general expression for $g_0(\omega, \eta)$. This may now be written as in (5)

$$g_0(\omega, \eta) = \sum_{\nu=0}^{\infty} (i\omega)^{2\nu} g_{02\nu}(\eta). \tag{12}$$

The coefficients of the development of $f_0^{(\lambda_0)}(\omega, \eta)$, obtained from (11) by putting $\lambda_1 = \lambda_2 = \cdots = 0$, are called $f_{02\nu}^{(\lambda_0)}(\eta)$. The coefficients $g_{02\nu}(\eta)$ then have to be calculated successively from a relation

$$g_{02\nu}(\eta) = \frac{1}{q}\left[h_0(\eta) f_{02\nu}(\eta) - h_0(\eta) \sum_{i=1}^{\nu} f_{02i}^{(\lambda_0)}(\eta)\, g_{02(\nu-i)}(\eta)\right], \tag{13}$$

$$(\nu = 1, 2 \ldots), \ (g_{00}(\eta) = 1).$$

At first sight this does not look encouraging, but it is simpler than one might think and an actual computation using (13) is not necessary. In fact, $f_{02\nu}(\eta)$ and $f_{02\nu}^{(\lambda_0)}(\eta)$ contain according to (11) and (9) the factor $1/h_0(\eta)$. But even in $h_0(\eta) f_{02\nu}(\eta)$ $(\nu > 0)$ there are terms with a factor $1/h_0(\eta)$, and in $h_0(\eta) f_{02\nu}^{(\lambda_0)}(\eta)$ all terms (for $\nu > 0$) contain this factor, which is directly seen from I (2). On the other hand, it was proved in connexion with the discussion of (4) that $g_0(\omega, \eta)$ does not contain $h_0(\eta)$ at all. Therefore in (13) all functions of the right member except the first one have the factor $1/h_0(\eta)$. As the remainder after the subtraction of these functions from $h_0(\eta) f_{02\nu}(\eta)$ is a function without this factor, $g_{02\nu}(\eta)$ is obtained from the function $\dfrac{h_0(\eta)}{q} f_{02\nu}(\eta)$ by cancelling precisely all terms with the factor $1/h_0(\eta)$. Therefore we only have to make some slight changes in (11) to have the general formula for $g_{02\nu}(\eta)$. All functions $D_{0jk\ldots}(\eta)$ are multiplied by $h_0(\eta)/q$ and in the summation in (9) all terms containing a factor $1/h_0(\eta)$ are excluded. More completely, we introduce

$$E_{jk\ldots}(\eta) = \sum_{\substack{n_j=0 \\ n_k=0 \\ \vdots}}{}' \frac{\dfrac{\lambda_{n_j}}{\lambda_0} \cdot \dfrac{\lambda_{n_k}}{\lambda_0} \cdots}{h_j(\eta + d_{n_j})\, h_k(\eta + d_{n_j} + d_{n_k}) \ldots}, \tag{14}$$

where $\sum'$ means that terms with $1/h_0(\eta)$ have to be excluded, or, observing the first number in the row of indices which is zero, $j\,k\ldots r\,s\,0\,t\ldots$, the terms corresponding to $n_j = n_k = \cdots = n_r = n_s = n_0 = 0$. Then we have from (11)

$$(15)\qquad g_{02\nu}(\eta) = r_1 \, {}_{\nu-1}\!\left[\sum_{m_{\nu-1}=1}^{2} r_{m_{\nu-1}} \, {}_{\nu-2}\!\left[\sum_{m_{\nu-2}=1}^{m_{\nu-1}+1} r_{m_{\nu-2}} \cdots {}_{1}\!\left[\sum_{m_1=1}^{m_2+1} r_{m_1} Q_1^{m_2+1-m_1} \right]_1 \cdots \right.\right.$$

$$\left.\left. \cdots Q_{\nu-2}^{m_{\nu-1}+1-m_{\nu-2}} \right]_{\nu-2} Q_{\nu-1}^{2-m_{\nu-1}} \right]_{\nu-1} E(\eta)_{1_{\nu}\dot{-}1\ldots\nu-1\,i\,\nu\,\nu-1\ldots 210}, \qquad (\nu = 1, 2, \ldots).$$

We have, for example

$$(15')\qquad g_{00}(\eta) = 1, \quad g_{02}(\eta) = r_1 E_{10}(\eta),$$

$$g_{04}(\eta) = r_1 [r_2 E_{1210}(\eta) + r_1 E_{1010}(\eta)], \text{ etc.} \qquad (r_1 = 1/3,\ r_2 = 4/15, \ldots).$$

In general, the main part of the expression for a function for a certain ν can be calculated from the expression for $\nu - 1$ according to the following formula, which is useful in actual calculations. For that purpose we first define as a generalization of $g_{02\nu}(\eta)$ a function $g_{02\nu,\,ij\ldots rs}(\eta)$ which is obtained formally by putting the indices $i\,j\ldots r\,s$ before the indices of all the terms $E_{tu\ldots}(\eta)$ in the expression calculated for $g_{02\nu}(\eta)$. It is then easily seen that

$$(16)\qquad g_{02\nu}(\eta) = r_1 g_{02(\nu-1),\,10}(\eta) + r_2 g_{02(\nu-1),\,12}(\eta) +$$

$$+ r_1 r_2 r_3 \, {}_{\nu-3}\!\left[\sum_{m_{\nu-3}=1}^{4} r_{m_{\nu-3}} \, {}_{\nu-4}\!\left[\sum_{m_{\nu-4}=1}^{m_{\nu-3}+1} r_{m_{\nu-4}} \cdots Q_{\nu-4}^{m_{\nu-3}+1-m_{\nu-4}} \right]_{\nu-4} Q_{\nu-3}^{4-m_{\nu-3}} \right]_{\nu-3}$$

$$E(\eta)_{123_{\nu}\dot{-}3\ldots\nu\dot{-}1\,;\,\nu\ldots 210}, \qquad (\nu = 2, 3 \ldots).$$

This simplifies the actual calculation of $g_{02\nu}(\eta)$ considerably, though even here the operators Q occur. But in (16) the number of successive operations is two less than in (15). For $\nu = 2$ the last term vanishes, and we get the expression in (15′); for higher values we have, for example

$$(16')\qquad g_{06}(\eta) = r_1 g_{04,\,10}(\eta) + r_2 g_{04,\,12}(\eta) + r_1 r_2 r_3 E_{123210}(\eta), \ \ldots$$

Some analytic properties of $f_0(\omega, \eta)$

Nothing has been said, so far, of the range of validity of the development (12). Taking $Re\{\eta\}$ larger than the largest of the numbers $1 - d_\nu$ ($\nu = 1, 2 \ldots$), it is easily proved that (12) is convergent for sufficiently small values of $|\omega|$. By means of analytic continuation $g_0(\omega, \eta)$ then may be calculated in larger domains of the η-plane and the ω-plane as long as no singularities are met with.

In the constant λ case we have for $f_0^{(\lambda_0)}(\omega, \eta)$, except for the poles, only the fixed singularities at $\omega = \pm i$ in the ω-plane. Taking first the case $Re\{\eta\} \to \infty$, remembering that we then have $h_n(\eta) \to 1$, it is seen directly from (9) that the variable λ case is obtained from the constant λ case by substituting $\lambda(E_0) = \lambda_m = \sum \lambda_\nu$ for λ_0, giving then $g_0(\omega, \eta)$ singularities at $\omega = \pm i\frac{\lambda_0}{\lambda_m}$, as appears immediately from (5), (9) and (11). But $g_0(\omega, \eta)$ is singular at these points and regular in the interval between them $\left(\text{if } \frac{\lambda_0}{\lambda_m} \leq 1\right)$ even for finite η-values. This may be shown from the original equations (2) as has been done by WICK [15] by moving step by step to the left in the η-plane, expressing $f_n(\omega, \eta)$ by means of the values of these functions at the points $\eta + d_k$, $\eta + d_k + d_l, \ldots$ $(k, l, \ldots = 1, 2, \ldots)$

For all values of $\theta = \eta - 1$ in the right halfplane of θ, avoiding in the ω-plane only the points $\omega = \pm\omega_\nu(\theta), \pm\omega_\nu(\theta + d_k), \pm\omega_\nu(\theta + d_k + d_l), \ldots (k, l \ldots = 1, 2 \ldots)$, (and $\omega = \pm i$), where $\pm\omega_\nu(\theta)$ denote the different poles of $f_0^{(\lambda_0)}(\omega, \theta)$, $g_0(\omega, \theta)$ is seen to be regular in this way, except for the fixed singularities at $\omega = \pm i\frac{\lambda_0}{\lambda_m}\cdot$ The points avoided are the poles of $f_0(\omega, \theta)$, of which all except $\omega = \pm\omega_0(\theta)$ are poles of $g_0(\omega, \theta)$. For this function $\omega = \pm\omega_0(\theta)$, in particular, is a regular point as well as the point $\theta = 0$ in the θ-plane. Therefore, for the θ-values quoted, the expression (15) for $g_0(\omega, \theta)$ may be analytically continued in all directions of the ω-plane as long as these poles and the points $\pm i\frac{\lambda_0}{\lambda_m}$, $\pm i$ are avoided.

The position of the singularity nearest $\omega = 0$ determines the radius of convergence of (12). Now the values of importance here are real values of θ, making all poles arising from $\pm\omega_0$ lie between $= -i$ and $= +i$, and purely imaginary values of ω in the interval between the fixed singularities. These are therefore of importance only for $\lambda_m > \lambda_0$, which also is obvious for physical reasons. Now let us consider the frequently occurring special case when all $\lambda_\nu > 0$. Then for θ real ≥ 0 and purely imaginary ω, the power series of $g_0(\omega, \theta)$ contains only positive terms $(g_{02\nu}(\theta) > 0)$. The singularities of the series therefore are situated at the points of intersection of the circle of convergence with the imaginary axis, and must be either a pole singularity belonging to $\pm\omega_0(\theta + d_m)$ (d_m = the smallest of the numbers $d_1, d_2, \ldots$) or the fixed singularities $\pm i\frac{\lambda_0}{\lambda_m}$, depending on which is nearest $\omega = 0$. Except when d_m and the λ variation are very small, taking a fixed real value for $\theta \geq 0$, the poles will lie outside the circle of radius λ_0/λ_m, and then the radius of convergence of (12) is exactly $\frac{\lambda_0}{\lambda_m}\cdot$ This is always valid, if θ is large enough. For the purpose of integration the primary development (12) is sufficient in this case, as it is useful for all values of the variables needed.

In the general case with arbitrary values of λ_ν it is difficult to predict anything about the circle of convergence of (12) for real $\theta \geq 0$, as the coefficients may have arbitrary signs, and it probably has to be estimated numerically in each special case. Possibly we then have to perform an analytic continuation to the point $i\lambda_0/\lambda_m$ (if $\lambda_0/\lambda_m < 1$).

3. Calculation of the neutron density at small and intermediate distances when $\lambda(E)$ varies slowly with E

When $\lambda(E)$ is a slowly varying function the evaluation of (3) may be done in quite the same way as in I, as long as r is not too large. The result is a function $F_0(r, E)$ containing corrections to the age-approximation, which, so far, is the only general expression known for the neutron density for small r.

As in I, we may perform one of the integrations by the method of residues. If we integrate ω in this way, we have to consider all the poles mentioned before and to sum the corresponding residues. But in restricting ourselves to the slow neutron density, it is sufficient here, as in I, to take into account only the main pole $\omega = \omega_0(\theta) = i B(\theta)$. It is true that we now have many new poles, contained in $g_0(\omega, \theta)$, but these also are all easily seen to be "secondary". In fact, even the contribution from the pole which is the most critical one $\omega = \omega_0(\theta + d_m)$ is seen from (2) to be only of the magnitude $\exp(-d_m x)$ as compared to the main pole term. If we suppose $x \gtrsim 10$, for example, this is sufficient for neglecting it, as d_m in general is of the magnitude ≈ 1.

Calculating now the residue of $f_0(\omega, \theta)$ at $\omega = \omega_0(\theta)$, and writing it, as in I, for small $|\theta|$ as a power series in θ, this series is obtained in the following way. It contains first the factor resulting from $f_0^{(\lambda_0)}$ determined by I (13), (14) and (16). In this paper we shall use for I (14) the notation $A_0(\theta)$, and we write $\alpha_{\nu 0}$ for the coefficients of its power series in θ which are calculated by I (16). Then we have the factor resulting from g_0, which is obtained by development of $g_{02\nu}(\theta)$ around $\theta = 0$, and inserting in (12) $\omega = i B(\theta)$, which gives the function $G_0(\theta)$

$$G_0(\theta) = g_0[\omega_0(\theta), \theta] = 1 + g_1 \theta + g_2 \theta^2 + \cdots \tag{17}$$

Here $g_1, g_2 \ldots$ are numerical constants, but contain all λ_ν as parameters. The main numerical difficulties in solving the problem with λ variable lie in this computation of g_ν and they increase rapidly with increasing values of ν, and of course also with increasing number of parameters λ_ν which are used in the formula (1) for $\lambda(E)$. Concerning the function $G_0(\theta)$ we know that it must have a singularity for the value of θ giving $B(\theta) = \lambda_0/\lambda_m$. The root of this equation (for $\lambda_0/\lambda_m < 1$) is called θ_m. On the other hand, this must be the first singularity on the real positive θ-axis, as the substitution $\omega = \omega_0(\theta)$ in $g_0(\omega, \theta)$ makes all the singularities deriving from the poles of $g_0(\omega, \theta)$ move farther to the right. Therefore $G_0(\theta)$ may be calculated in the interval $0 \leq \theta < \theta_m$ on the real axis, either directly by the series (17) or by making an analytic continuation of the series in addition. In the special case $\lambda_\nu > 0$, if the λ variation is not too small, we have the radius of convergence θ_m for (17) directly.

We now define

$$A(\theta) = A_0(\theta) G_0(\theta), \tag{18}$$

given for small $|\theta|$ by a development

$$A(\theta) = 1 + \alpha_1 \theta + \alpha_2 \theta^2 + \cdots, \tag{19}$$

just as in I (14), and for other values of θ as the analytic continuation of this series. The formula which results for $F_0(r, E)$ is then I (17) or I (36) (after division with λ_0^3) and the remaining integration in θ (or φ) may be performed by the saddle-point method as in I. As soon as $\lambda_0/\lambda_m < 1$, however, there is an important restriction in the applicability of the formulas of I, dependent on the new singularities at $= \pm i\frac{\lambda_0}{\lambda_m}$ of $f_0(\omega, \theta)$ in this case (or, physically, dependent on the importance of the primary neutrons when their mean free path is larger than that of the slow neutrons). In fact, I (17) and I (36) are now valid only if the contribution to the integration in ω from the integral around the point $\omega = + i\frac{\lambda_0}{\lambda_m}$ along a cut in the imaginary axis (similar to the cut from $\omega = + i$ in I) can be neglected.

But this is possible only when the real values of θ of importance in the evaluation of I (17), i.e. the saddle-point θ_s, is considerably smaller than θ_m. This also appears directly from the fact that $A(\theta_s)$ in this case is singular for $\theta_s = \theta_m$. When θ_s is near θ_m the singularity at $\omega = + i\frac{\lambda_0}{\lambda_m}$ gives the essential contribution to the integral and the evaluation of (3) is then possible only with a close knowledge of the nature of this singularity. Now if r and λ_m/λ_0 are not large θ_s may be taken as in I as the root of the equation $x - R B'(\theta) = 0$, and therefore $R/x = r/\lambda_0 x \ll 1/B'(\theta_m)$ is the condition for I (17), I (36) and the formulas derived from them to be useful. The extension of this domain depends essentially on the value of λ_m/λ_0. If this quantity is not too small (say $\gtrsim 2$), the condition is roughly $r \ll 2x\lambda_0^2/C^2\lambda_m$, obtained by taking the first term of I (11). In many cases, slowing down in water, for example, this is a very small distance, and other methods of calculation must be used.

When r is sufficiently below the limit above, we may take over directly the main formula I (40), divided by λ_0^3, for $F_0(r, E)$, and also I (41), (42), (43) and (44). Now for very small r we must of course expect these formulas somehow to contain the age-approximation. They also give the expected Gaussian behaviour, but in the same form as in I, $\exp(-r^2 C^2/4\lambda_0^2 x)$, whereas the age-approximation for the variable λ case contains instead of λ_0 a mean value $\bar{\lambda}$, I (49), or in this case with the λ-variation (1) and for sufficiently large x given by

$$\bar{\lambda}^2 x = \lambda_0^2 x' = \lambda_0^2 (x + a) = \lambda_0^2 \left[x + \sum_{\substack{k=0\\ l=0}}{}' \frac{\frac{\lambda_k}{\lambda_0}\cdot\frac{\lambda_l}{\lambda_0}}{(d_k + d_l)} \right] \tag{20}$$

$$(k, l = 0, 1, 2, \ldots, \; k = l = 0 \text{ excluded}),$$

formally corresponding to an increase of x to $x' = x + a$.

In order to solve this apparent inconsistency, we must first examine the approximations made in age-theory. This theory is better the larger M is, and in fact may be considered as an asymptotic theory for $M \to \infty$. In this limit the age-theory formula must follow from the integral (3). Following now the evaluation of (3) to the next step, I (17), and making $M \to \infty$, this solves the problem. For in this limit $A_0(\theta) = 1$, corresponding to the constant λ case, but the new factor $G_0(\theta)$ of $A(\theta)$ in the variable λ case, has non-vanishing coefficients in the limit $M \to \infty$. This depends on the limiting form of $h_0(\theta)$, $h_0(\theta) \to 2\,\theta/M$ for $M \to \infty$ (but $h_n(\theta) \to 1$ for $n \geq 1$), making the term $r_1^{\nu} E_{101010\ldots}(\theta)$, which is the largest one in $g_{02\nu}(\theta)$, behave like M^{ν}, whereas $\omega_0^2(\theta) \to -6\,\theta/M$, so that $g_{02\nu}(\theta) \neq 0$. Indeed, $G_0(\theta)$ has the limiting form $\exp(a\,\theta)$ for $M \to \infty$, which follows because the age-approximation then is exact, remembering (20) and I (17). Of course this can also be proved by a direct calculation of $G_0(\theta)$, taking into account the facts mentioned above, but the proof is not very simple and the reader will be spared it.

Now we can make use of this property of $G_0(\theta)$ even for finite M. It is natural to separate the factor $\exp(a\,\theta)$ from $G_0(\theta)$, writing

$$(21) \qquad G_0(\theta) = e^{a\,\theta}\, G_0'(\theta), \quad A(\theta) = e^{a\,\theta}\, A'(\theta).$$

The new functions may be determined for small $|\,\theta\,|$ by the series

$$(22) \qquad G_0'(\theta) = 1 + g_1'\,\theta + g_2'\,\theta^2 + \cdots, \quad A'(\theta) = 1 + \alpha_1'\,\theta + \alpha_2'\,\theta^2 + \cdots,$$

and these coefficients are obtained from g_{ν} and α_{ν}, computed before (17), (19), in developing $\exp(a\,\theta)$. Then we can still use I (17), only substituting x' for x and $A'(\theta)$ for $A(\theta)$, and similarly in all the expressions evaluated for $F_0(r, E)$. Now, this is advantageous in every respect. Firstly the age-theory formula now appears as the first approximation of our expression for $F_0(r, E)$, and the next approximations contain in a natural way the corrections to age-theory. On the whole, we obtain an extension of the domain of applicability of our formulas in this way. For the coefficients α_{ν}' are smaller than α_{ν}, at least for ν small, which is helpful in developing $F_0(r, E)$. The increase of x to $x' = x + a$ (for $a > 0$, which is always valid in practice) is also advisable in this respect, as $1/x'$ is essentially the parameter of these developments. This increase is also important because it decreases the saddle-point θ_s, which extends the formula to larger values of r. For θ_s is now determined by the equation $x' - RB'(\theta_s) = 0$ (for r and λ_m/λ_0 not too large), similar to I (33), and the maximum value of r by the condition $R/x' = r/\lambda_0\, x' \ll 1/B'(\theta_m)$.

We may now take over all the formulas I (17) and I (33), (34) . . . (44), changing x to x' and $A(\theta)$ to $A'(\theta)$. It seems advisable to reproduce here the simplified form of I (40) for small R/x', as in this case, when α_{ν}' in general is not small or decreasing with increasing ν (in contrast to the constant λ case), the developments for $F_0(r, E)$ as given in the earlier papers mentioned [5], [6] are best written in a somewhat different way. We obtain, either by a partial development in I (40), or directly from I (36)

$$(23)\qquad F_0(r, E) =$$

$$= \frac{Q\cdot 3\,h_1}{4\,\pi^{3/2}\,\lambda_0^2\,E\,r\,(x'\,\tau)^{1/2}}\,e^{x'\theta_s - R\,B(\theta_s)}\left\{U(\varphi_s)\left[1+\frac{\delta_2}{2}-\frac{\delta_1^2}{4}+\frac{3\,\delta_2^2}{8}+\frac{3\,\delta_1\gamma}{4}+\cdots\right]+\right.$$

$$+\frac{U'(\varphi_s)}{(x'\,\tau)^{1/2}}\left[\frac{\delta_1}{2}-\frac{3\,\gamma}{4}+\frac{3\,\delta_1\,\delta_2}{4}\cdots\right]-\frac{U''(\varphi_s)}{2\,x'\,\tau}\left[\frac{1}{2}+\frac{3\,\delta_2}{4}+\cdots\right]-$$

$$\left.-\frac{U'''(\varphi_s)}{6\,(x'\,\tau)^{3/2}}\left[\frac{3\,\delta_1}{4}+\cdots\right]+\frac{U^{\mathrm{IV}}(\varphi_s)}{24\,(x'\,\tau)^2}\left[\frac{3}{4}+\cdots\right]+\cdots\right\}.$$

Here θ_s and $\varphi_s = +\sqrt{\theta_s}$ are the saddlepoints in the θ- and φ-planes, respectively, given together with τ by I (37) ($x \to x', R = r/\lambda_0$). The exponent in (23) then has the form

$$(24)\qquad x'\,\theta_s - R\,B(\theta_s) = -\left[\frac{R^2\,C^2\,(1+2\,\tau)}{3\,x'\,(1+\tau)^2}+R\,\delta(\varphi_s)\right],$$

$$\delta(\varphi) = -C\,(\beta_2\,\varphi^5+\beta_3\,\varphi^7+\beta_4\,\varphi^9+\cdots).$$

Further we have in (23)

$$(25)\qquad U(\varphi) = \varphi\,A'(\varphi^2) = \varphi+\alpha_1'\,\varphi^3+\alpha_2'\,\varphi^5+\cdots,$$

and

$$(26)\qquad \delta_n = \frac{R}{(x'\,\tau)^{n/2}}\,\frac{\delta^{(n)}(\varphi_s)}{n\,!},\quad (n = 1, 2, \ldots)\qquad \gamma = \delta_3 - \frac{R\,C\,\beta_1}{(x'\,\tau)^{3/2}}.$$

In (23) the term $-R\,\delta(\varphi_s)$ in the exponent, as well as the terms δ_n and γ, are only small correction terms, negligible for sufficiently small R/x'. Then we also have $\tau \approx 1$, and (24) transforms into the Gaussian exponent $-R^2\,C^2/4\,x'$. In this small domain around the source (roughly for $R/x' \ll 1/C\sqrt{3\,\beta_1}$), (23) may be simplified (see earlier paper [5], formula (31), taking $x \to x'$ and $\alpha_\nu \to \alpha_\nu'$). Now in (23) we must always have $\theta_s \ll \theta_m$, implying an upper limit for R/x', but also, for the validity of I (37), roughly $R/x' < 2\sqrt{2}/C\sqrt{3\,\beta_1}$. Of these two conditions the first one is the more restrictive when λ_m/λ_0 is not near unity. Then (23) is the only simplified form of I (40) which is applicable to the variable λ case. For a very small λ variation, (23) is completed at larger distances, compatible with the condition $\theta_s \ll \theta_m$, by the approximate I (44), with θ_s taken from I (38) (making the usual changes). The condition for all these formulas to be useful is, as usual, that the first terms of the developments decrease sufficiently rapidly.

The corrections to age-theory appear in (23) partly by the deviation of (24) from the value $-R^2\,C^2/4\,x'$, and partly by the deviation of the development in (23) with the main term $\frac{1}{\sqrt{\tau}}\left\{U(\varphi_s) - \frac{U''(\varphi_s)}{4x'\tau} + \frac{U^{\mathrm{IV}}(\varphi_s)}{32(x'\tau)^2} - \cdots\right\}$ from the value $RC/2x'$. These deviations both increase rapidly with increasing R/x'. The first one is of the same kind as in the constant λ case and is also the more important for a slow variation of λ. The second one also appears in the constant λ case though it is then of little importance. In this case, however, it is of dominating significance as soon as the λ variation is not very small. For a large variation of λ, in particular for small M, the transformation (21) is not sufficiently effective. Then the development of (23) or I (40) may be useless even for $r = 0$, which indicates that age-theory is a very poor approximation in this case.

These formulas for the neutron density, giving the corrections to age-theory, have been derived for the case of a slow variation of λ. They are applicable only up to a certain distance r, determined by the condition $\theta_s \ll \theta_m$, depending physically on the increasing importance at larger distances of the source neutrons with the mean free path $\lambda_m > \lambda_0$.. If λ_m is not much larger than λ_0, implying a sufficiently low value of E_0, and when λ_0 and M are not small, these formulas are sufficient for practical purpose. Otherwise, it may be necessary to complete them. This will be done next, though we are restricted then to the practically most important case of a mean free path that decreases with decreasing energy of the neutrons.

4. Calculation of the neutron density when $\lambda(E)$ decreases with decreasing E

New information about $f_0(\omega, \theta)$

The extension of the formulas for the neutron density to larger distances, or to the case of a rapidly varying mean free path, requires knowledge of the kind of singularity possessed by $f_0(\omega, \theta)$ at $\omega = +i\lambda_0/\lambda_m$ (for $\lambda_0 < \lambda_m$), (or of $G_0(\theta)$ at $\theta = \theta_m$). This seems to be difficult to obtain in the general case. However, in the case where $\lambda(E)$ decreases with decreasing E, WICK in an important paper [15], has succeeded in calculating the behaviour of $f_0(\omega, \theta)$ at this point, and has thereby derived the asymptotic dependence of $F_0(r, E)$ on r and x for $r \to \infty$. Using here this information about $f_0(\omega, \theta)$, and combining it with our previous knowledge of this function and also making use of analytic continuation, we will now calculate the neutron density at distances outside those already attained. In the special case $r \to \infty$, the formula then obtained automatically shows the asymptotic dependence predicted by WICK.

Now, in this case we get from the integration in ω in (3) an essential contribution from the cut along the imaginary axis from the point $\omega = +i\lambda_0/\lambda_m$. Therefore it is convenient to take instead the opposite order of integration, calculating first the integral in θ by the method of residues. The poles of $f_0^{(\lambda_0)}(\omega, \theta)$ then are the inverse functions of $\omega = \omega_\nu(\theta)$, say $\theta = \theta_\nu(\omega)$, i.e. $\theta = \theta_\nu[\omega_\nu(\theta)]$, in particular for the main pole $\theta = \theta_0(\omega)$. Further we make a slight change of variables in (3), putting

$$\omega = ik, \tag{27}$$

and denote the main pole henceforth as $\theta_0(k)$. This is obtained by taking the inverse function of $k = B(\theta)$ or $k^2 = [B(\theta)]^2$, and can be calculated for small $|k|$ as a development in k^2, from I (11)

$$\theta_0(k) = k^2/C^2 + b_1 k^4/C^4 + \cdots, \tag{28}$$

and for larger k from I (27)

$$\theta_0(k) = \left[\frac{1}{\theta_1} + \frac{k-h_0}{h_1} - \frac{h_2(k-h_0)^2}{h_1^3} + \cdots\right]^{-1}. \tag{29}$$

When the values of the coefficients in I (11) and I (27) are known numerically, the coefficients of (28) and (29) may also be rapidly computed numerically. These developments are also of use in the previous calculations for the value of θ_m $(k = \lambda_0/\lambda_m)$. We also point out here the simple approximate formula for $\theta_0(k)$ contained in I (32), valid for θ_0 real, positive and not too small.

The result of the integration in θ is then easily obtained, neglecting as before the contribution from all "secondary" poles. Denoting simply by $A(k)$ the function resulting from (18), (19), after substitution of (28) for θ, where $A(k)$ is given for small $|k|$ as a development

$$A(k) = 1 + a_1 k^2 + a_2 k^4 + \cdots, \tag{30}$$

we have

$$F_0(r, E) = \frac{Q \cdot 3h_1}{8\pi^2 \lambda_0^2 E r} \frac{1}{i} \int_{L_k} dk\, e^{-kR + x\theta_0(k)} A(k)\, \theta_0'(k). \tag{31}$$

This formula is similar to I (17) and I (36), and may also be evaluated by the saddle-point method. The path of integration L_k, which originally is the imaginary axis, may be transformed to pass through a saddle-point k_s along a path of steepest descent. When $A(k)$ can be considered slowly variable, k_s may be taken as the root of the equation $x\theta_0'(k) - R = 0$, corresponding to $x - RB'(\theta) = 0$ before, having the value $k_s = B(\theta_s)$. With the simple approximate values $B_s(\theta)$ of $B(\theta)$, I (31) and I (32) k_s is equal to $B_s(\theta_s)$, giving

$$k_s = \frac{2RC^2(2+\tau)}{3x(1+\tau)^2} \quad \text{and} \quad k_s = a - \sqrt{\frac{bx}{R}}. \tag{32}$$

The first equation is obtained from I (31), (37), and is valid for small R/x; the second is obtained from I (32), (38) and holds for larger R/x.

But these values of k_s are useful only for sufficiently small r and for λ_0/λ_m near unity. Then (31) may be evaluated in quite the same way as I (17) or I (36) and gives equivalent developments for $F_0(r, E)$. The same transformation as in (21) may naturally also be used to extend the range of validity to larger distances. However, we are going to use (31) only when the previous formulas for $F_0(r, E)$ are insufficient, and the transformation (21) must be replaced by a more powerful one.

Now WICK has shown, that for $\theta \to \infty$ to the right in the θ-plane, f_0 behaves like $(1 - k\,\lambda_m/\lambda_0)^{\varrho-2}$ for $k \to \lambda_0/\lambda_m$, where ϱ is a constant < 2. Here ϱ may be determined as the lowest eigenvalue of a certain Schrödinger equation (WICK [15], eq. (105), cf. also Appendix in the present paper), of which the higher eigenvalues give additional terms to the term quoted of the same type as this one, although these are of vanishing importance at $k = \lambda_0/\lambda_m$. Then we can move step by step to the left in the θ-plane, solving (2) successively for decreasing values of θ, as was described before. The result is that as long as the poles are avoided, f_0 behaves in the same way for $k \to \lambda_0/\lambda_m$ even for finite θ-values.

This singularity of course belongs to $g_0\,(i\,k, \theta)$, which shows this behaviour in particular for real $\theta \geq 0$. Then, however, we may have a pole in the k-plane between the origin and the fixed singularity if θ is small. But after inserting $\theta = \theta_0\,(k)$ in $g_0\,(i\,k, \theta)$, all these poles move beyond $k = \lambda_0/\lambda_m$, and the resulting function $g_0\,(i\,k, \theta_0\,(k))$ is regular in the interval between $k = 0$ and this point. With the variation of $\lambda\,(E)$ now assumed, in any case if all $\lambda_\nu > 0$, it was proved before that (12) has the radius of convergence λ_0/λ_m when θ is real and chosen sufficiently to the right in the θ-plane. This is valid also after the substitution $\theta = \theta_0\,(k)$, if the variation of λ is not too small. But this function $g\,(i\,k, \theta_0\,(k))$, given by a development in k^2, multiplied with a similar development of $A_0\,(\theta_0\,(k))$, resulting from the residue of the factor $f_0^{(\lambda_0)}$, is precisely $A\,(k)$ (30). Thus $A\,(k)$ is regular in particular for $0 \leq k < \lambda_0/\lambda_m$. Further the development (30) of $A\,(k)$ in general converges within the whole of this interval, if λ_0/λ_m is not too small. For $k \to \lambda_0/\lambda_m$, $A\,(k)$ behaves like $(1 - k\,\lambda_m/\lambda_0)^{-\alpha}$, taking $\alpha = 2 - \varrho$ ($\alpha > 0$). We therefore put

$$(33)^{1} \qquad A\,(k) = \frac{L\,(k)}{(1 - k\,\lambda_m/\lambda_0)^{\alpha}} = \frac{P\,(k)}{[1 - (k\,\lambda_m/\lambda_0)^2]^{\alpha}},$$

where the second expression is taken because $A\,(k)$ is an even function.

$L\,(k)$ and $P\,(k)$ are obtained as power series in k. We know that for $0 \leq k < \lambda_0/\lambda_m$ in particular, $L\,(k)$ (and $P\,(k)$) is regular, and that it is continuous also at $k = \lambda_0/\lambda_m$. We cannot assume regularity at this point too, but only that $L\,(k)$ consists of a regular part + terms singular like $(1 - k\,\lambda_m/\lambda_0)^{\varrho_i - \varrho}$, containing the other eigenvalues $\varrho_i\,(i = 1, 2\,\ldots)$. However, these terms all vanish at $k = \lambda_0/\lambda_m$. Then to the first approximation, the behaviour of (33) around this point is determined by knowledge of the value $L\,(\lambda_0/\lambda_m) = L_m$. This value may be obtained from the power series in k for $L\,(k) = A\,(k)$ $(1 - k\,\lambda_m/\lambda_0)^{\alpha}$ (or for $P\,(k)$) and then either by putting $k = \lambda_0/\lambda_m$ directly in this series or by first using analytic continuation. For this purpose various methods may be devised, and an account of this will be given later in connexion with the reproduction of the numerical applications. The essential fact here is that such a calculation always is allowed because $L\,(k)$ is continuous at λ_0/λ_m and is possible in practice by computation of a number of coefficients $a_1, a_2, \ldots$ of $A\,(k)$. Now, it seems in general that for the next lowest eigenvalue ϱ_1 we have $\varrho_1 - \varrho > 1$. Then the behaviour of $A\,(k)$ at the singularity may be obtained to the next approximation, too, calculating graphically, for

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[1] Similar transformations (with $a = 1$) can also be used for the study of $\theta^0\,(k)$ (28) at the singularity $k = 1$ (cf. I).

example, the value $L'_m = L'(\lambda_0/\lambda_m)$ and developing the regular part of $L(k)$. The graphical study of $L(k)$ also gives a rough estimation of the coefficient l_m of $(1 - k\,\lambda_m/\lambda_0)^{\varrho_1 - \varrho}$ in the complete development of $L(k)$ around $k \quad \lambda_0/\lambda_m$.

Calculation of $F_0(r, E)$

Making the substitution (33) in (31), and putting

$$K(k) = L(k)\,\theta'_0(k), \quad T(k) = P(k)\,\theta'_0(k), \tag{34}$$

we now proceed with the evaluation of (31). When r is not too large, we take the singularity at λ_0/λ_m into account best by writing it as a logarithm in the exponential (cf. BAERWALD [2]). Thus we have for the integrand in (31)

$$T(k)\,e^{\Psi(k)}, \text{ with } \Psi(k) = -k\,R + x\,\theta_0(k) - \alpha \log\left[1 - (k\,\lambda_m/\lambda_0)^2\right]. \tag{35}$$

This transformation is similar to (21) but more effective because it sums up the main singular part of $A(k)$. We have used here the second transformation (33), because $P(k)$ is an even function ($T(k)$ odd) and easy to handle numerically. Its power series converges satisfactorily at $k = \lambda_0/\lambda_m$ and may be used directly without any preliminary treatment.

We may now use a saddle-point integration in (31). But instead of (32) which is restricted to small r or λ_m/λ_0, we take k_s as the root of the equation (at least approximately)

$$\Psi'(k) = -R + x\,\theta'_0(k) + \frac{2\,k\,\alpha\,(\lambda_m/\lambda_0)^2}{1 - (k\,\lambda_m/\lambda_0)^2} = 0. \tag{36}$$

L_k is chosen as a straight line through k_s parallel to the imaginary k-axis. The resulting expression for $F_0(r, E)$ is obtained and may be written essentially in the same way as the general formula I (40)

$$F_0(r, E) = \frac{Q \cdot 3\,h_1}{(2\sqrt{\pi})^3\,\lambda_0^2\,r\,E}\,d\,e^{\Psi(k_s)}\left\{M_0\left[1 - \cdots\right] + \cdots\right\}. \tag{37}$$

Here the development is exactly the same as in I (40), and is therefore not written out. The quantities d, ε, m_i and n_i in this development now are abbreviations of

$$d = \left[\Psi''(k_s)/2\right]^{-1/2}, \quad \varepsilon = d\,\Psi'(k_s), \tag{38}$$

$$m_i = d^i\,\frac{T^{(i)}(k_s)}{i!} \quad (i = 0, 1, \ldots), \qquad n_{i-2} = -\,d^i\,\frac{\Psi^{(i)}(k_s)}{i!} \quad (i = 3, 4, \ldots).$$

M_i is obtained by I (41). If the saddle-point k_s is determined exactly, which then in general has to be done numerically, we have $\varepsilon = 0$ and $M_i = m_i$. Otherwise we may take

$$k_s = \frac{R\,C^2}{2\,x''}, \quad x'' = x + \alpha\,(C\,\lambda_m/\lambda_0)^2, \tag{39}$$

as long as this makes ε small (i.e. for sufficiently small r). This gives for $F_0(r, E)$ a Gaussian main term for small r, but with a coefficient of $-r^2$ in the exponent different from that of the age-approximation ($x' \to x''$).

(37) is our main formula for $F_0(r, E)$. It is applicable for much larger values of r and λ_m/λ_0 than (23). As in I (40) ε ought to be small, and m_i, M_i and n_i are regarded as small quantities of the ith order of magnitude ($\sim 1/(\sqrt{x})^i$). The development in (37) as in I (40) is essentially an asymptotic series in $1/\sqrt{x}$, containing only even powers, which are calculated up to the fourth order. The formula thus contains three successive approximations of $F_0(r, E)$. The zeroth approximation is obtained by taking only the term M_0 in the development, the second approximation by adding

$$-M_0\,(15\,n_1^2/16 + 3\,n_2/4) - 3\,M_1\,n_1/4 - M_2/2 \text{ to this term (cf. I (40)).}$$

The complete development calculated in (37) and I (40) gives the fourth approximation. Then we can get an account of the accuracy of (37) (as of I (40) before) by studying the values of the successive approximations. This has to be done numerically in each special case. Then it appears that the simple zeroth approximation already gives comparatively good values. The second approximation in general changes the value by about 10–50 %, whereas the fourth approximation in general implies a change of a few % only. Although the fourth approximation may be needed for great accuracy, the second approximation therefore ordinarily is sufficient in practice. Due to a considerable cancellation of terms, this also holds when the order of magnitude relations for each n_i and m_i are no longer strictly valid, which is the case for increasing r and λ_m/λ_0. In fact, a check of (37) with numerical integration of (31) along L_k shows an excellent agreement even in such cases. At the same time, however, (39) no longer makes ε small and is therefore useless, and we have no simple general relation $k_s = k_s(R, x, \lambda_m/\lambda_0)$, although it may be possible to calculate for given values of x and λ_m/λ_0. Thus in a small intermediate domain we cannot obtain the general analytic dependence of F on R, x and λ_m/λ_0, though (37) of course still is useful by giving first certain values of k_s within this domain and determining then the corresponding values of r from (36). For still larger r, when $k_s \to \lambda_m/\lambda_0$ we could again derive a simple approximate analytic expression for k_s by developing $\Psi(k)$ around $k = \lambda_m/\lambda_0$. However, although (37) even for $r \to \infty$ shows the correct dependence on r and x (cf. formula (49)), the development then seems to become uncertain and it is convenient to modify the method of integration as will be described below. We may also expect (37) to begin to fail in accuracy for increasing values of λ_m/λ_0, roughly when $(C\,\lambda_m/\lambda_0)^2$ is of the magnitude $\gtrsim x$. Possibly numerical integration of (31) along L_k then will be needed for more accurate values of F_0.

For large values of r we are faced with the problem of a saddle-point integration when the saddle-point approaches a singularity of the integrand. Then it is convenient to develop around the point of singularity instead of around the saddle-point. In fact, problems of a similar kind have been treated before, by BAERWALD [2] for example, though in the case of a single pole. Here the singularity is a branch-point, too, which necessitates some changes of the method. We take in (31)

$$k = \lambda_0/\lambda_m + \psi, \tag{40}$$

and develop $\theta_0(k)$

$$\theta_0(k) = \theta_m + \psi\,\theta'_m + \psi^2\,\theta''_m/2 + \cdots, \qquad \theta_m^{(i)} = \theta_0^{(i)}(\lambda_0/\lambda_m). \tag{41}$$

According to (36), the condition for k_s to be near λ_0/λ_m then can be approximately written $R - x\,\theta'_m \gg \alpha\,\lambda_m/\lambda_0$. Then we use the first transformation (33), multiply according to (34) with the derivative of (41), taking the following approximate expressions of $L(k)$ and $K(k)$

$$L(k) = L_m + L'_m\,\psi + l_m(-\psi\,\lambda_m/\lambda_0)^{\varrho_1-\varrho}, \tag{42}$$

$$K(k) = K_0 + K_1\,\psi + K_2\,\psi^2 + k_0(-\psi)^{\varrho_1-\varrho}.$$

Here $L_m \ldots$ are the constants mentioned before, the new constants $K_0 \ldots$ are calculated from them by (34) and (41). K_2 is not completely determined in this way, but still we can take it as a very rough estimation. Now (31) takes the form

$$F_0(r,E) = \frac{Q\cdot 3\,h_1}{8\,\pi^2\,\lambda_0^2\,E\,r}\left(\frac{\lambda_0}{\lambda_m}\right)^{\alpha} e^{x\,\theta_m - r/\lambda_m}\,\frac{1}{i}\int\limits_{L_\psi} d\psi\, e^{-(R-x\,\theta'_m)\,\psi + \frac{x\,\theta''_m}{2}\psi^2 + \cdots}\cdot \tag{43}$$

$$\cdot\,(-\psi)^{-\alpha}\,(K_0 + K_1\,\psi + K_2\,\psi^2 + k_0(-\psi)^{\varrho_1-\varrho}).$$

L_ψ is the path of integration in the ψ-plane, cut along the real axis from 0 to ∞. L_ψ may be chosen for convenience so as to make the integrand in (43) decrease as fast as possible from $\psi = 0$, only subject to the restriction that it always lie to the left of the cut. This choice will depend essentially on which of the two main terms of the exponent, $-(R - x\,\theta'_m)\,\psi$ or $\psi^2\,x\,\theta''_m/2$ is the dominating one. The first one decreases more rapidly for ψ increasing along the real axis (since $R - x\,\theta'_m > 0$), the second one for ψ leaving the origin along the imaginary axis, as we always have $\theta''_m > 0$. Therefore two cases can be distinguished

1) $\alpha\,\lambda_m/\lambda_0 \ll R - x\,\theta'_m \lesssim \sqrt{x\,\theta''_m/2}$.

This requires a sufficiently large value of θ''_m, say $\theta''_m \gtrsim 10$. Therefore this case applies only to moderate values of λ_m/λ_0. We take L_ψ as a straight line to the left of, but infinitesimally close to the imaginary ψ-axis. The terms in $\psi, \psi^3, \psi^4, \ldots$ of the exponent are then expanded, and the result is a development containing integrals of the type $1/i\int_{L_\psi} d\psi\, e^{\beta\,\psi^2}(-\psi)^{-\varkappa}$ $(\beta = x\,\theta''_m/2)$. These integrals are connected with Γ-functions in a very simple way. This is seen in detail by the transformation $\psi = - z^{1/2}$, determining the radical by a cut through the negative real z-axis. Then the integral above transforms into a well known contour-integral for the Γ-function, and has the value

$$\pi \beta^{\frac{\varkappa-1}{2}} \Big/ \Gamma\left(\frac{\varkappa+1}{2}\right).$$

We obtain the following expression for F_0

$$(44) \qquad F_0(r, E) =$$

$$= \frac{Q \cdot 3 h_1}{8 \pi \lambda_0^{\varrho} \lambda_m^{2-\varrho}} \frac{e^{x \theta_m - r/\lambda_m}}{E r} \left(\frac{x \theta_m''}{2}\right)^{\frac{1-\varrho}{2}} K_0 \left\{ \Omega_{\varrho-1} - \frac{1}{\sqrt{x \theta_m''/2}} \left[\frac{\theta_m''' \Omega_{\varrho+2}}{3 \theta_m''} + \frac{K_1 \Omega_\varrho}{K_0}\right] + \right.$$

$$+ \frac{1}{x \theta_m''/2} \left[\frac{\theta_m^{\mathrm{IV}} \Omega_{\varrho+3}}{12 \theta_m''} + \frac{\theta_m'''^{2} \Omega_{\varrho+5}}{18 \theta_m''^{2}} + \frac{K_1 \theta_m''' \Omega_{\varrho+3}}{3 K_0 \theta_m''} + \frac{K_2 \Omega_{\varrho+1}}{K_0}\right] +$$

$$\left. + \frac{1}{(x \theta_m''/2)^{\frac{\varrho_1-\varrho}{2}}} \frac{k_0 \Omega_{\varrho_1-1}}{K_0} + \cdots \right\}.$$

Here we have

$$(45) \qquad \Omega_{-\gamma} = \frac{p_\gamma(\varepsilon)}{\Gamma\left(\frac{\gamma+2}{2}\right)} + \frac{q_{\gamma-1}(\varepsilon)}{\Gamma\left(\frac{\gamma+1}{2}\right)}, \quad p_\gamma(\varepsilon) = 1 + \frac{\varepsilon^2}{2!} \frac{\gamma}{2} + \frac{\varepsilon^4}{4!} \frac{\gamma(\gamma-2)}{2^2} + \ldots,$$

$$q_\gamma(\varepsilon) = \varepsilon + \frac{\varepsilon^3}{3!} \frac{\gamma}{2} + \frac{\varepsilon^5}{5!} \frac{\gamma(\gamma-2)}{2^2} + \ldots, \quad \varepsilon = (R - x \theta_m') / \sqrt{x \theta_m''/2}, \quad (R = r/\lambda_0).$$

A condition for case 1) is $\varepsilon \lesssim 1$, but, in fact, (44) is applicable also for slightly larger ε, if only the terms of the development in $1/\sqrt{x \theta_m''/2}$ in (44) decrease sufficiently rapidly. This main condition for the formula to be useful is fulfilled for $x \gg 1$ and $\theta_m'' \gg 1$, the latter of which restricts the applicability of (44) to moderate values of λ_m/λ_0.

$$\textbf{2)} \quad (R - x \theta_m') \begin{cases} \gg \alpha \lambda_m/\lambda_0 \\ > \sqrt{x \theta_m''/2}. \\ > K_1/K_0 \end{cases}$$

In this case the path of steepest descent from $\psi = 0$ lies near the real positive ψ-axis for small $|\psi|$. For increasing values of $Re\{\psi\}$, however, this path leaves the real ψ-axis and passes to infinity roughly parallel to the imaginary axis. Therefore we can still take for one part L_s of L_ψ two straight lines parallel to the imaginary axis, starting from points infinitesimally close to a point $k = k_s'$ in the interval $\lambda_0/\lambda_m < k_s' < 1$, one on each side of the cut.

A convenient choice for k_s' is the value given by (32), for then the exponential part of the integrand decreases as fast as possible from $k = k_s'$ along L_s. The other part of L_ψ, L_c, joins these points with each other. L_c is a contour along the cut, infinitesimally close to it, which encloses the origin in the ψ-plane ($k = \lambda_0/\lambda_m$). Thus we have in (43) $\int\limits_{L_\psi} = \int\limits_{L_s} + \int\limits_{L_c}$. In this sum the last term, the "branch-point" contribution, always dominates the first term, because it contains the main term e^{-r/λ_m}, whereas $\int\limits_{L_s}$ behaves like $e^{-R} = e^{-r/\lambda_0}$ for $r \to \infty$. For the calculation of $\int\limits_{L_c}$ the terms in $\psi^2, \psi^3, \ldots$ are expanded, and then we obtain a sum of integrals of the type $\frac{1}{i}\int\limits_{L_c} d\psi\, e^{-\gamma\psi}(-\psi)^{-s}$ $(\gamma = R - x\,\theta_m')$. Now such an integral, where the path of integration is extended to $\psi = +\infty$ in both directions, is precisely the contour integral for the Γ-function encountered above. From this value we must here subtract the contributions from the paths between the point $k = k_s'$ and the real, positive infinity. Then the integral quoted has the value $2\pi(\gamma)^{s-1}\nu(s,u)$, where

$$\nu(s,u) = \frac{1}{\Gamma(s)}\left[1 - \frac{Q(1-s,u)}{\Gamma(1-s)}\right]. \tag{46}$$

Here $Q(1-s,u)$ is an incomplete Γ-function

$$Q(1-s,u) = \int\limits_u^\infty dt\, e^{-t} t^{-s}, \quad \text{with} \quad u = (k_s' - \lambda_0/\lambda_m)(R - x\,\theta_m'). \tag{47}$$

Using these notations, (43) takes the form

$$\begin{aligned} F_0(r,E) = \frac{Q \cdot 3 h_1}{4\pi\lambda_0^\varrho \lambda_m^{2-\varrho}} \frac{e^{x\theta_m - r/\lambda_m}}{E r} (R - x\,\theta_m')^{1-\varrho} K_0 \Bigg[\nu(2-\varrho,u) - \\ - \frac{K_1}{K_0(R - x\,\theta_m')}\nu(1-\varrho,u) + \frac{\left(\frac{K_2}{K_0} + \frac{x\,\theta_m''}{2}\right)\nu(-\varrho,u)}{(R - x\,\theta_m')^2} - \\ + \frac{k_0\,\nu(2-\varrho_1,u)}{K_0(R - x\,\theta_m')^{\varrho_1-\varrho}} + \cdots \Bigg] + F_{0s}(r,E). \end{aligned} \tag{48}$$

Here $F_{0s}(r,E)$ is the contribution to the integral from L_s and may be calculated essentially as a saddle-point integral (if k_s' is not too close to λ_0/λ_m). It contains the same main term as I (40), $\exp(-R k_s' + x\,\theta_0(k_s'))$ and has the character of a correction term in (48). It will be negligible when u is sufficiently large. At the same time the incomplete Γ-function (47), of magnitude $\exp(-u)$, can be dropped in (46). Actually it seems possible in most cases to use (37) or (44) and (45) up to a distance where u attains a value say ≈ 5. If we use (48) for larger distances only, we can substitute $1/\Gamma(s)$ for $\nu(s, u)$ in (48) and drop the term $F_{0s}(r, E)$, which simplifies the formula considerably. Possibly there will be a small intermediate domain where neither (37) and (44) nor (48) will give accurate values of $F_0(r, E)$. Then we can interpolate the neutron density at these distances with sufficient accuracy for practical purpose.

The asymptotic formula. Discussion of $F_0(r, E)$

We notice that for very large distances (48) is simplified, giving the following expression

$$F_0(r, E) = \frac{Q}{4\pi\lambda_m^3}\,\mu\,\frac{e^{x\theta_m - r/\lambda_m}}{E\,(r/\lambda_m)^{\varrho}}, \qquad \mu = \frac{3\,h_1\,\lambda_m\,K_0}{\lambda_0\,\Gamma(2-\varrho)}. \tag{49}$$

(49) is the complete formula for the asymptotic slow neutron density. It shows the dependence of $F_0(r, E)$ on x and r predicted by WICK [15]. Further it contains the constant μ, depending on M and on the rapidity of the λ-variation, and which can be determined numerically as has already been described. But from (48) we also obtain direct information on how large r-values we have to take for the asymptotic formula to be valid. The condition is $r/\lambda_0 \gg x\,\theta'_m$, completed for large λ_m/λ_0 with $R \gg K_1/K_0$, which essentially implies $r \gg \lambda_m$ (a similar, but weaker condition must be fulfilled also for (48) to be valid). For $r/\lambda_0 \ll x\,\theta'_m$ we found before that the roughly Gaussian behaviour of (37) was dominating. For increasing r, when $r/\lambda_0 - x\,\theta'_m \gg \alpha\,\lambda_m/\lambda_0$, it appears from (44) and (48) that we have already the same main term $\exp(x\,\theta_m - r/\lambda_m)$ in the expression for the neutron density as we have in the asymptotic formula, though multiplied with a different factor. But this shows anyhow that even at these distances the rapid source neutrons of mean free path λ_m are of dominating importance for the slow neutron density, even if the complete asymptotic formula is not valid before the limit $\lambda_0\,x\,\theta'_m$ (and λ_m) has been considerably surpassed.

In a discussion of the neutron density the value of $\lambda_0\,x\,\theta'_m$ is a characteristic quantity. In general, θ'_m (as well as θ_m) decreases with increasing values of λ_m/λ_0 (M = constant), but increases with increasing values of M (λ_m/λ_0 = = constant). This gives the general result, to be expected of course for physical reasons too, that large values of x, λ_0, M, and small values of λ_m/λ_0 (near to 1) all are factors which extend the validity of the roughly Gaussian behaviour and make the domain of the asymptotic formula and the related formulas (48) and (44) apply to larger values of r. If λ_m/λ_0 is not too small (roughly $\gtrsim 2$), this appears directly from the approximate value of $\lambda_0\,x\,\theta'_m \approx$ $\approx 2\,x\,\lambda_0^2/\lambda_m\,C^2$, for large M further simplified to $M\,x\,\lambda_0^2/3\,\lambda_m$ (I (45)). As was mentioned before, this is quite a small distance for water, for example, which means that (44), (48) and even (49) have great practical importance in this case. But for carbon, for example, this quantity is much larger, which then makes formula (37), (23) or I (44) most useful, though even here (44) and (48) are applicable at distances of practical interest. These facts will appear in detail later in connexion with the reproduction of the numerical results.

A discussion of the neutron density as complete as in I is more difficult in this case, but the most important facts have been given above. For small r-values and a slow variation of λ, the discussion in I on the whole is applicable ($x \to x'$). But for larger values of r and λ_m/λ_0, there are also striking differences between the constant λ case and the case treated here. If we take a constant value of x and study the space distribution, this may be described as in I as a quasi-exponential decay, though now with the decay-length decreasing towards λ_m. But this behaviour is now in general more marked and begins earlier than in the constant λ case. As it has been pointed out before [15], the energy distribution of neutrons at a certain distance r has quite another form here than in I for large r-values, as it tends to a limit independent of r for $r \to \infty$. This also implies that the energy distribution function is not so rapidly decreasing with increasing values of E as in I. Its dependence on M and on λ_m/λ_0 is determined essentially by θ_m and the variation of θ_m was quoted above. Thus for large λ_m/λ_0 we have roughly $\theta_m \approx (\lambda_0/\lambda_m C)^2$.

Further, we have in (44), (48) and (49) the constants ϱ and μ (K_0). For a roughly twofold variation of λ and a moderate M-value ϱ is near unity. ϱ increases for increasing λ_m/λ_0 and varies only very slowly with M. Finally μ increases with increasing M, roughly as M for large M, and with increasing λ_m/λ_0; it tends to a limit independent of λ_m/λ_0 for large values of this quantity. These facts appear in detail from the numerical results.

5. Applications

Methods of numerical calculations

A numerical calculation of the first corrections to the age-approximation, by means of a computation of g_1 and g_2, for example, can always be made without undue labour, even if several terms are used to describe the function $\lambda(E)$ (1). On the other hand, it is natural to restrict the number of terms in (1) so as not to be unduly large, giving an unnecessarily refined formula for $\lambda(E)$. We have to take into consideration the uncertainties of the experimental measurements of $\lambda(E)$, for instance. Further, as we are now concerned with larger values of E_0 than in the constant λ case, other general assumptions made, like considering only S-scattering between the neutrons and the nuclei, are not fulfilled with the same accuracy. Anyhow, we must expect our formulas for the neutron density to have less precision now than for those small values of E_0 for which I was applicable, and therefore it is not worth while taking too many terms in (1). In this paper we have used for the numerical calculations merely a two-term variation of $\lambda(E)$, capable of describing a linear, and also a more general kind of monotonic λ variation

$$\lambda(E) = \lambda_0 + \lambda_1 \left(\frac{E}{E_0}\right)^d. \tag{1'}$$

In many cases of practical interest this is sufficiently accurate.

The numerical calculations at first aim at a computation of the coefficients $g_1, g_2, \ldots$ of $G_0(\theta)$ in (17) or $a_1, a_2, \ldots$ in (19). Besides, the knowledge of

$B(\theta)$ and $\theta_0(k)$ is essential, but this has been obtained before in I, as well as the factor $A_0(\theta)$ in (18). $G_0(\theta)$ is the result of the substitution $\omega = iB(\theta)$ in the series (12) for $g_0(\omega, \theta)$ $(\theta = \eta - 1)$ and a development of the coefficients $g_{02\nu}(\theta)$ around $\theta = 0$ $(\eta = 1)$. The main numerical difficulties lie in this calculation of $g_{02\nu}(\theta)$. Now, in the case treated here, the summation in (14) contains only terms with $n_j, n_k, \cdots = 0$ or 1. Any function $E_{jk\ldots}(\theta)$ with ν indices $jk\ldots$ occurring in $g_0(\omega, \theta)$ can therefore be written as a polynomial of degree ν in λ_1/λ_0

$$E_{jk\ldots}(\theta) = \sum_{\varkappa=1}^{\nu} (\lambda_1/\lambda_0)^{\varkappa} E_{jk\ldots}^{(\varkappa)}(\theta). \tag{50}$$

Here the "components" $E_{jk\ldots}^{(\varkappa)}(\theta)$ of $E_{jk\ldots}(\theta)$ $(\varkappa = 1, 2 \ldots \nu)$ depend only on θ. In the future the value of this component, given as a development in θ around the origin will be denoted by $E_{jk\ldots}^{(\varkappa, 0)}$. What we finally want is thus the developments $E_{jk\ldots}^{(\varkappa, 0)}$ of the components of all the functions $E_{jk\ldots}(\theta)$ occurring in the expressions evaluated for $g_{02\nu}(\theta)$, (15). These coefficients may also be written as in (50)

$$g_{02\nu}(\theta) = \sum_{\varkappa=1}^{2\nu} (\lambda_1/\lambda_0)^{\varkappa} g_{02\nu}^{(\varkappa)}(\theta). \tag{51}$$

When the component $g_{02\nu}^{(\varkappa)}(\theta)$ $(\varkappa = 1, 2 \ldots 2\nu)$ is developed around the origin the resulting power series in θ is written $g_{02\nu}^{(\varkappa, 0)}$.

For the basic functions $h_i(\theta)$ in all these calculations the numerical method of obtaining developments around $\theta = 0$ was given in I, Appendix B (for $M > 1$). As these could be calculated rather rapidly up to a high degree in θ they can be used even to get the developments around the points $\theta = \mu d$ of $h_i(\theta)$, though of course only up to a lower power in θ. This has actually been done in the case of carbon. For $M = 1$ the original formula for $h_i(\theta)$, which is simple in this case, had to be used, I (4) and (5). Finally we invert these series, calculating the developments of $1/h_i(\theta)$ around $\theta = \mu d$ $(\mu = 0, 1, 2, \ldots)$, and denote this by $1/h_i^{(\mu)}$ (i and μ not both $= 0$).

If more than the first or the two first terms in (17) are wanted, it seems best to use the following recurrence relation for the calculations, obtained directly from (14).

$$E_{ijk\ldots}^{(\varkappa, \mu)} = \frac{1}{h_i^{(\mu)}} E_{jk\ldots}^{(\varkappa, \mu)} + \frac{1}{h_i^{(\mu+1)}} E_{jk\ldots}^{(\varkappa-1, \mu+1)}, \qquad (\varkappa, \mu = 0, 1, 2, \ldots) \tag{52}$$

with the initial conditions $E_0^{(0, \mu)} = 1/h_0^{(\mu)}$, $E_0^{(1, \mu)} = 1/h_0^{(\mu+1)}$, defining $\frac{1}{h_0^{(0)}} = 0$. Here a generalization $E_{jk\ldots}^{(\varkappa, \mu)}$ of the function $E_{jk\ldots}^{(\varkappa, 0)}$, defined before, has been introduced. This is the coefficient of $(\lambda_1/\lambda_0)^{\varkappa}$ in the expression (14), when this coefficient is developed in a power series in θ around the point $\theta = \mu d$ $(\eta = 1 + \mu d)$, $(\mu = 0, 1, 2, \ldots)$, with the summation in $n_1, n_2 \ldots$ extended (for $\mu > 0$) to *all* combinations of the indices of summation. Thus for the function $E_{jk\ldots}^{(\varkappa, \mu)}$, given as a power series in θ, the restriction concerning the summation in (14) has to be used only for $\mu = 0$. But in computing all the functions $E^{(\varkappa,0)}_{ijk\ldots}$ which enter into the functions $g_{02\nu}(\theta)$, successively by means of (52), this restriction is taken into account automatically by the formal requirement $1/h^{(0)}_0 = 0$.

Now, when the basic developments in θ $1/h^{(\mu)}_i$ $(i, \mu = 0, 1, \ldots)$ have been calculated, we also know the initial quantities $E^{(\varkappa,\mu)}_0$ $(\varkappa = 0, 1, \mu = 0, 1 \ldots)$ for the application of (52) (because the last index in all functions $E_{jk\ldots}(\theta)$ in $g_{02\nu}(\theta)$ always is zero). We compute first $E^{(\varkappa,\mu)}_{10}$ and then we have to calculate $E^{(\varkappa,\mu)}_{210}$, $E^{(\varkappa,\mu)}_{010}$ $(\varkappa = 0 \ldots 3, \mu = 0, 1, \ldots)$. We could go on in this way until all the developments $E^{(\varkappa,0)}_{jk\ldots}$ wanted have been obtained, and then insert them into the expressions evaluated from (15). But instead we define developments $g^{(\varkappa,\mu)}_{02\nu}$, obtained by substituting $E^{(\varkappa,\mu)}_{jk\ldots}$ for $E^{(\varkappa,0)}_{jk\ldots}$ everywhere in the expression for $g^{(\varkappa,0)}_{02\nu}$; and similarly we define functions $g^{(\varkappa,\mu)}_{02\nu,ij\ldots}$. Then the calculations may be done more rapidly by making use of (16). In fact, for the formation of these functions or, on the whole, any function which is a linear expression in the quantities $E^{(\varkappa,\mu)}_{ij\ldots}$ a formula, completely analogous to (52), holds. $g^{(\varkappa,\mu)}_{04}$, for example, (15′), is obtained by applying this formula to the development of $r_1(r_2 E^{(\varkappa,\mu)}_{210} + r_1 E^{(\varkappa,\mu)}_{010})$, with $i = 1$. For the computation of $g^{(\varkappa,\mu)}_{06}$, (16′), we then use this formula to get $r_1 g^{(\varkappa,\mu)}_{04,0}$ and $r_2 g^{(\varkappa,\mu)}_{04,2}$ and apply it once again to the sum of these quantities and calculate $r_1 g^{(\varkappa,\mu)}_{04,10} + r_2 g^{(\varkappa,\mu)}_{04,12}$. Then we only have to add $r_1 r_2 r_3 E^{(\varkappa,\mu)}_{123210}$, which must be computed separately, using (52). For higher functions $g^{(\varkappa,\mu)}_{02\nu}$ $(\nu = 4, 5 \ldots)$ (16) is used and the main part of such a function is rather rapidly computed in the same way as above from the values of the immediately preceding term. The rest of formula (16), the operator expression, which remains to be evaluated, causes a little more trouble, but even here several terms can be put together to form expressions $g^{(\varkappa,\mu)}_{02\nu,ij\ldots}$ derived from lower values of ν.

The numerical difficulties are considerably greater here than in the case of constant λ, and it does not seem possible to compute as many coefficients a_ν (g_ν) as in I. But fortunately, this is not necessary, even if we want the same accuracy as in I, except when the λ variation is so small that we are near the limiting case $\lambda = \lambda_0 = \text{constant}$. For when $\lambda_1 > 0$ and λ_1/λ_0 is not too small, the development (30), which has to be analytically continued to $k = \lambda_0/\lambda_m = \lambda_0/(\lambda_0 + \lambda_1)$, already has this point lying on the circle of convergence, which is advantageous compared with the cases of analytic continuation met with in I. But the number of coefficients may be reduced also because the same accuracy as in I, for example, is now useless for reasons mentioned above. In fact, the actual calculations, including the cases of carbon and water, were performed only up to g_5, but this seemed to be sufficient for practical purposes. Then we had to start with the computation of the values of the functions $h_i(\theta)$ and their first p derivatives at the points $\theta = \mu d$ for all values of i, p and μ $(\mu \geq 0)$ compatible with the equations $\mu + i + 2p \leq 10$ and $i + p \leq 5$ (in general: $\leq 2n$, and $\leq n$, respectively, if n coefficients g_ν are wanted). In this way the basic quantities $1/h^{(\mu)}_i$ were obtained as power series in θ up to θ^p. $E^{(\varkappa,\mu)}_{10}$ $(\varkappa = 0, 1, 2)$ is then calculated for $\mu = 0, 1 \ldots 8$, $E^{(\varkappa,\mu)}_{210}$ and $E^{(\varkappa,\mu)}_{010}$ $(\varkappa = 0, \ldots 3)$ for $\mu = 0, 1 \ldots 7$ etc. Each step in these computations consists of the formation of the sum of two products, each containing two factors in the form of power series (52). The coefficients of the resulting power series are then sums of a number of products and are rapidly obtained numerically by the use of an electric computing machine.[1] This computation is quite analogous to the numerical evaluations of equations I (21), . . . (25) and is performed in practice by the same methods. The numerical calculations which remain for obtaining $g_1 \ldots g_5$ and $\alpha_1 \ldots \alpha_5$, when all $g_{02\nu}^{(\varkappa, 0)}$ ($\varkappa = 1, 2 \ldots 2\nu$, $\nu = 1, 2 \ldots 5$) finally are known, are also of the same type.

In the same way as in (51) α_ν can be written as a polynomial of degree 2ν in λ_1/λ_0

$$\alpha_\nu = \sum_{\varkappa=0}^{2\nu} \alpha_{\nu\varkappa} (\lambda_1/\lambda_0)^\varkappa. \tag{53}$$

The values of the constants $\alpha_{\nu\varkappa}$ are the final results of the first and the largest part of the numerical computations. Of course explicit formulas of the first few of these constants, containing the values of $h_i(\theta)$ and its derivatives at $\theta = 0, d, 2d, \ldots$, can be derived. This was done in an earlier paper [5] for α_1 and α_2. It is also possible to calculate for α_ν developments in $1/M$, valid for large M, similar to I (45), (46). Thus we have, for example

(54)[2]
$$\alpha_{11} = \frac{2}{d}\left[1 + \frac{5\,d}{3\,M} + \cdots\right], \quad \alpha_{12} = \frac{1}{2\,d}\left[1 + \frac{10\,d}{3\,M} + \cdots\right],$$

$$\alpha_{21} = -\frac{8}{15\,d\,M} + \cdots, \quad \alpha_{22} = \frac{2}{d^2}\left[1 + \frac{103\,d}{15\,M} + \cdots\right], \quad \alpha_{23} = \frac{1}{d^2}\left[1 + \frac{49\,d}{5\,M} + \cdots\right],$$

$$\alpha_{24} = \frac{1}{8\,d^2}\left[1 + \frac{208\,d}{15\,M} + \cdots\right].$$

But these developments are not at all as useful as those for C, $\beta_1, \beta_2, \ldots$ and $\alpha_{10}, \alpha_{20}, \ldots$ given in I, because the convergence is much poorer and is still worse for increasing values of ν and $\varkappa$. (54) shows the expected asymptotic behaviour for $M \to \infty$, remembering that we now have $a = \dfrac{2}{d}\dfrac{\lambda_1}{\lambda_0} + \dfrac{1}{2\,d}\left(\dfrac{\lambda_1}{\lambda_0}\right)^2$, (20), (21).

Proceeding now with the calculations, we obtain from $B(\theta)$ the coefficients of (28) for $\theta_0(k)$ and insert them in (19). Instead of α_ν we then have coefficients a_ν (30), also containing λ_1/λ_0 as parameter, but which could also be written as in (53). Now, if λ_1/λ_0 is not small, both α_ν and a_ν will have large values, increasing rapidly with ν, and therefore it is convenient for the numerical computations to take a new variable t in (30), writing $C(t)$ for $A(k)$.

$$k = t\,\lambda_0/\lambda_m = t\,\lambda_0/(\lambda_0 + \lambda_1), \quad C(t) = 1 + c_1 t^2 + c_2 t^4 + \cdots, \quad c_\nu = a_\nu \left(\frac{\lambda_0}{\lambda_m}\right)^{2\nu}. \tag{55}$$

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[1] *Facit*, mod. ESA, Åtvidaberg, Sweden.

[2] These formulas may easily be derived also for the general case (1).

Then the series for $C(t)$ has the radius of convergence $= 1$, at least when λ_1/λ_0 is not small. In the actual calculations this was confirmed by the facts that the c_ν were close to unity, and the power series for $C(t)$ is of the same type as the simple series $1 + t^2 + t^4 + \cdots$.

For the calculation of $K_0, K_1 \ldots$, we now first compute the eigenvalue $\varrho\,(= 2 - \alpha)$ using the variation method with a linear variation function (Appendix). We could then, according to (33) consider the function $L(t\lambda_0/\lambda_m) = C(t)(1-t)^\alpha$, which is continuous also at $t = 1$, and try to calculate this function and its derivative at this point. But the power series in t for this function has alternating signs with the coefficients near unity, of the type $1 - t + t^2 - \cdots$. It is true that such a series can be summed for $t = 1$ if it is first conveniently transformed, using "Euler's transformation" for example.[1] This was also actually performed in some cases. But there are simpler methods available for this computation. The reason why the development of $C(t)(1-t)^\alpha$ has the inexpedient form quoted, is that this function is still singular and infinite at $t = -1$, as $C(t)$ is an even function of t. Therefore it is much better instead to study the development in t of the inverted function $(1-t)^{-\alpha}/C(t)$, which is finite and continuous for $-1 \leq t \leq 1$. This series ought to be of the same type as $1/(1 - t + t^2 \ldots) = 1 + t$, and it was actually found that the coefficient of t was near $+1$, whereas the other coefficients were smaller and decreasing. The convergence of this series seemed to be satisfactory for $t = \pm 1$, so that it could be used for determination of the functional values at these points. Then we also have a certain control of the convergence of the series from the sum at $t = -1$, which must be zero.

This way of calculating L_m and K_0 seemed to be the best one, but a third, and still simpler method was also used, and the values resulting from the three methods were compared. This makes use of the development in t^2 for the function $P(t\lambda_0/\lambda_m) = C(t)(1-t^2)^\alpha$, which is continuous for $-1 \leq t \leq +1$. This is rapidly computed, as only even powers of t occur, and the series showed satisfactory convergence at $t = \pm 1$ (it ought to be of the type $(1 + t^2 + t^4 + \cdots)(1 - t^2) = 1$, which also was confirmed). The sum at $t = 1$ divided by 2^α gives L_m (and K_0). The graphical study of $L(t\lambda_0/\lambda_m)$ in the neighbourhood of $t = 1$ then gives a rough value of L'_m (and K_1), and finally in this way an estimation can be made of l_m (and k_0).

Application to carbon

This case was treated in I for λ constant, and for the values of $B(\theta)$ and $A_0(\theta)$ we refer to Tables 3 and 4 in this paper. According to the measurements of the neutron cross section for carbon [1], [4], the treatment in I covers all values of E_0 roughly up to 0.1 Mev. For higher E_0, roughly up to 2.5 Mev, the following expression gives a very good approximation of $\lambda(E)$

(56) $\lambda(E)_{\mathrm{cm}} = 2.65 + 2.35\,E$ (Mev), thus $d = 1$, $\lambda_0 = 2.65$, $\lambda_1 = 2.35\,E_0$ (Mev).

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[1] See HARDY, "Divergent series", Oxford 1949, (1, 3, 3), and (1, 3, 4), page 7, observing that in the latter formula $b_n = \Delta^n a'_0$, if $a'_n = (-1)^n a_n$. ($\Delta^n a'_0 = \Delta^{n-1} a'_1 - \Delta^{n-1} a'_0$)

Then we have taken the carbon density as 1.60 g/cm^3; for any other value, (56) of course has to be multiplied with a constant. First we give the expressions (28) and (29) of the inverse function $\theta_0(k)$ of $k = B(\theta)$, obtained from I,

$$(57) \quad \begin{cases} \theta_0(k) = 2.2214\,k^2 + 1.4116\,k^4 + 1.1556\,k^6 + 1.0372\,k^8 + 0.9741\,k^{10} + \cdots, \\ \theta_0(k) = (0.3 - 1.8500\,u + 1.897\,u^2 - 7.68\,u^3 - 5.0\,u^4 - 52\,u^5 - \cdots)^{-1}, \text{ with} \\ \quad u = k - 0.8326, \end{cases}$$

the first of which is used for insertion in (19) and besides for determination of θ_m roughly for $\lambda_0/\lambda_m \lesssim 0.6$, and the second used for θ_m for $\lambda_0/\lambda_m \gtrsim 0.6$. Then, however, I (32) and I, Table 4 give a simple approximate formula which also may be used. For the derivatives of $\theta_0(k)$ more accurate values can be obtained by differentiating the inverse function $k = B(\theta)$, using all the coefficients of I, Table 3, when θ_m has been determined before.

The result of the main part of the numerical calculations, the constants $a_{\nu\varkappa}$, is given in Table 1 (for $a_{\nu 0}$ see also I, Table 3).

Table 1

$a_{\nu\varkappa}$

$\varkappa \backslash \nu$	1	2	3	4	5
0	−0.30435	0.05258	−0.00520	0.00024	−0.0000
1	2.2722	−0.0532	0.0269	−0.0023	−0.000
2	0.6413	3.2809	0.8236	0.1476	0.012
3		1.9873	4.4416	2.098	0.576
4		0.3148	3.8588	6.221	3.801
5			1.2570	6.379	8.930
6			0.1441	3.057	9.946
7				0.7087	6.037
8				0.0646	2.069
9					0.381
10					0.029

Then, for every value of E_0 within the interval 0.1–2.5 Mev (roughly) we can obtain the basic constants $a_1 \ldots a_5$ by a very rapid computation according to (53) and (56). For low values of E_0 and small r, when (23) is applicable we can then first compute the constants a'_ν by (21) and (22), with $a = 2\,\lambda_1/\lambda_0 + \frac{1}{2}(\lambda_1/\lambda_0)^2$. For the high values of E_0 and for larger distances we transform (19) into (30), using (57), and after the calculation of $\varrho\,(a)$ we apply (33). Then we can use (37), (44) or (48), and some constants occurring in these formulas are given in Table 2 for three different E_0. Rough values of these constants for other E_0 may be obtained by interpolation, at least for ϱ, K_0, K_1, . . ., which are not easily computed directly.

In Table 2 the last, or possibly the two last figures of the values for ϱ have to be regarded as uncertain, as well as the last figure of K_0, μ and K_1. For K_2 and k_0 only one figure is significant. For the two highest values of E_0 we have, very roughly, $\varrho_1 - \varrho \approx 2$. Thus the term in k_0 in (44) and (49) is approximately

Table 2

E_0	θ_m	θ'_m	θ''_m	ϱ	K_0	K_1	K_2	k_0	μ
2	0.3155	1.916	7.39	0.7646	1.68	15	40	100	14.5
1	0.7691	3.628	14.1	0.4615	1.54	10	30	50	9.3
0.5	1.621	7.49	40.0	-0.1519	0.44				1.7

of the same importance as the term in K_2. For the lowest value of E_0 quoted an estimation shows that K_1 and K_2 are of the same magnitude as K_0. However, these values are omitted in the table because they are very rough; besides, they are of no practical importance in (44) or (49). As a demonstration of the calculations of $K_0, \ldots$ we take the value $E_0 = 2$ Mev. Then $C(t)$ (55) has the following development

$$C(t) = 1 + 1.6583\, t^2 + 2.178\, t^4 + 2.570\, t^6 + 2.870\, t^8 + 3.106\, t^{10} + \cdots.$$

Further we have

$$1/L(t\lambda_0/\lambda_m) = (1-t)^{-a}/C(t) = 1 + 1.2354\, t - 0.2776\, t^2 - 0.560\, t^3 - 0.141\, t^4 - \\ - 0.112\, t^5 - 0.016\, t^6 + 0.003\, t^7 + 0.003\, t^8 + 0.006\, t^9 + 0.001\, t^{10} + 0.000\, t^{11} + \cdots,$$

which converges well for $|t| = 1$, though a more accurate value of $\varrho(\alpha)$ probably would make the convergence still more rapid. It seems to give a reliable value of L_m (and K_0), and by a graphical study of the sum near $t = +1$, even of L'_m (and K_1 and K_2). The corresponding developments for the lower values of E_0 quoted show a somewhat slower convergence, but still seem useful. But as was pointed out above, for decreasing E_0 accuracy of the values $K_0, K_1, \ldots$ becomes less important.

Finally we study the validity of the formulas for $F_0(r, E)$ in this case. For $E_0 = 2$ (23) is not even applicable for $r = 0$, but instead (37) is useful up to fairly large distances. (48) is valid only when r has surpassed the value $5.08\, x$ cm sufficiently, but such a distance may still be of practical interest. The asymptotic formula (49) already begins to be approximately valid for $r > 30\, x$ cm, roughly. For the lower values of E_0 (23) may be used for small r, though (37) has a still larger range of applicability. The characteristic limit $\lambda_0\, \theta'_m\, x$ of r is larger now, $9.61\, x$ and $19.8\, x$ cm for $E_0 = 1$ and 0.5 Mev, respectively, which makes (44), (48) and (49) still less practically important in these cases. The asymptotic formula (49), for example, requires distances roughly $> 60\, x$ and $120\, x$ cm, respectively, to be approximately valid.

Application to water

The slowing-down of neutrons in water has been studied theoretically by several authors. FLÜGGE [3] has given a simplified treatment of this problem, which, however, is based on the diffusion equation and therefore must be considered as insufficient. VERDE and WICK [14] have developed a numerical method of calculation which is apt only for this case because it is based on the fact that λ_0 is small for water, which is taken into account by putting

$\lambda_0 = 0$ in (1′) ($d = 0.5$). But this approximation is very rough when the slow neutron density is wanted, even if it is possible to apply corrections afterwards. The application of the general treatment given here, with both terms in (1′) taken into account from the start, seems more reliable.

However, we have so far studied the slowing-down only in moderators containing one single element. Even if the oxygen atoms do not contribute very much to the scattering and slowing down in water, they ought to be taken into account, at least approximately. The simplest way of doing this is by assuming that the scattering cross sections of hydrogen and oxygen vary with E in the same manner, so that their ratio is constant and independent on E. Denoting as before the total mean free path in water as $\lambda(E)$, and the mean free path in hydrogen, i.e. in water, when only the scattering of the hydrogen atoms is taken into account, as $\lambda_H(E)$, we have $\lambda(E)/\lambda_H(E) = \varkappa =$ constant, and we only have to take everywhere in our formulas (cf. [13])

$$c_n(\eta) = \varkappa\, c_{nH}(\eta) + (1 - \varkappa)\, c_{n0}(\eta),$$

if $c_{nH}(\eta)$ and $c_{n0}(\eta)$ are the functions $c_n(\eta)$, I (5), (6) for hydrogen ($M = 1$) and oxygen ($M = 16$), respectively. In this sum the main contribution always comes from hydrogen, as $\varkappa$ is near unity, and in particular for $n > 0$, because then $c_{n0}(\eta)$ becomes very small compared to $c_{nH}(\eta)$.

Now, from the measurements of the cross sections [1], [4] we can take the following expression for the total mean free path (units for E and E_0 Mev)

$$(58)\qquad \lambda(E)_{\text{cm}} = 0.65 + 2.09\,(E)^{3/4},\ \text{thus}\ \lambda_0 = 0.65,\ d = 3/4,\ \lambda_1 = 2.09\,(E_0)^{3/4}.$$

Except when E is near a resonance value for the cross section of oxygen, this formula is quite a good approximation in the interval 0–2.5 Mev, and can be used even for higher energy values, though with less accuracy. Making a compromise between the values for small and large E, we can take, with good approximation in the interval quoted, $\varkappa = 0.87$.

First we have to calculate the function $B(\theta)$, using the methods developed in I. But since an approximation $\lambda = \lambda_0 =$ constant is not useful even for as low a value of E_0 as 0.01 Mev (giving roughly $\lambda_m/\lambda_0 = 1.1$), this function is not needed for large values of θ, and the analytic continuation to $\theta \to \infty$ with the determination of the constants of the asymptotic formula for constant λ is of very little practical use. Therefore, only the constants $3\,h_1$, C and β_n, I (11), h_n, I (27), and some values of a, b and c, I (32), were calculated. Thus we obtain

$$(59)^1\quad \left\{\begin{array}{l} 3\,h_1 = 1.244, \\ C = 1.04950,\ \beta_1 = 0.48408,\ \beta_2 = -0.26757,\ \text{etc.}\ \beta_3 \ldots \beta_{11} \\ 0.05819,\ 0.17818,\ -0.4556,\ 0.7920,\ -1.218,\ 1.781,\ -2.55,\ 3.64,\ -5.2, \\ \text{respectively, and} \\ h_0 = 0.660,\ h_1 = -0.127,\ h_2 = 0.035,\ h_3 = -0.009,\ h_4 = 0.002,\ \ldots \end{array}\right.$$

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[1] In these constants, as well as in other constants given, of course only two or three figures may be considered as physically correct. For mathematical reasons, however, more figures must be taken and treated as if they were correct.

The values h_n refer to a development I (27) around the point $y_1 = 1.5$, $\theta_1 = 2/3$, derived from the first development by means of $y_0 = 6$, $\theta_0 = 1/6$ as intermediate point. For the constants of I (32), used for the saddle-point θ_s in I (also useful for θ_m), we can take

$$a = 0.981,\ b = 0.360,\ c = 0.456,\ \text{for } 0.3 \lesssim \theta \lesssim 2.0,\ \text{or} \quad 0.50 \lesssim k \lesssim 0.83\ \ (1.6 \lesssim R/x \lesssim 17,\ \text{I }(38)). \tag{60}$$

The inverse function $\theta_0(k)$ of $k = B(\theta)$ has the following developments

$$\left\{\begin{aligned} \theta_0(k) &= 0.90789\,k^2 + 0.7980\,k^4 + 0.8270\,k^6 + 0.8071\,k^8 + 0.8261\,k^{10} + \cdots, \\ \theta_0(k) &= (1.5 - 7.89\,u + 17.0\,u^2 - 36\,u^3 + 60\,u^4 + \cdots)^{-1},\quad u = k - 0.660, \end{aligned}\right. \tag{61}$$

the first one used for the insertion in (19) and also for determination of θ_m, roughly for $\lambda_0/\lambda_m \lesssim 0.55$, the second one for θ_m for larger values of λ_0/λ_m roughly up to 0.8, which seems to be sufficient in practice. The first one may also be used for the derivatives for small λ_0/λ_m. These can also be calculated more accurately from $B(\theta)$, differentiating implicitly, as more coefficients were computed in (59).

Now we proceed to the constants $a_{\nu\varkappa}$. They are given in Table 3.

Table 3

$a_{\nu\varkappa}$

$\varkappa \backslash \nu$	1	2	3	4	5
0	−0.88984	0.71730	−0.4943	0.2442	0.016
1	3.8888	−0.6937	0.879	−1.069	1.33
2	1.1187	11.319	4.415	1.443	−1.84
3		7.2830	33.157	25.91	7.27
4		1.2175	31.455	101.76	100.6
5			10.923	117.84	319.3
6			1.3196	61.05	415.7
7				15.00	278.2
8				1.430	102.1
9					19.62
10					1.55

For very small values of E_0 the development in θ with the constants obtained from this table is useful directly. But in most cases the transformation to the variable k by means of (61) is needed, and the formula (37) for $F_0(r, E)$ is the most important one. Formula (48), however, is also of great practical interest in this case, much more than for carbon, for example. The same constants as for carbon above are given in Table 4 for four different values of E_0.

For the accuracy of the values of the constants ϱ, K_0 . . . we have the same limitations as for carbon. An estimation of the next lowest eigenvalue gives the very rough value $\varrho_1 \approx 2.5$ for all the values of E_0 quoted. The developments used for the determination of K_0, . . . show good convergence, at least

Table 4

E_0	θ_m	θ'_m	θ''_m	ϱ	K_0	K_1	K_2	k_0	μ
2	0.0226	0.296	2.06	1.0001	0.207	2.7	10	5	1.65
1	0.0538	0.477	2.44	0.9212	0.306	2.8	8	4	1.67
0.5	0.1197	0.781	3.38	0.7907	0.403	2.6	7	3	1.59
0.2	0.309	1.60	7.3	0.4841	0.42	2	6	6	1.16

for the three highest E_0. As must be expected in general, the lowest value of E_0 gives the poorest convergence. Therefore for $E_0 = 0.2$ the values of K_1, K_2 and k_0 are only very rough estimations.

As was mentioned above, formula (37) for $F_0(r, E)$ is the practically most important one in this case. In spite of the fact that water ought to be an unfavourable case for the application of our formulas on account of the particularly high values of λ_m/λ_0, (37) seemed to be more useful than one might expect. This appeared from a numerical study of the successive approximations of (37) and a check with numerical integration of (31). Using (37) and (48) (and for very small E_0 (44)), possibly combined with interpolation through a small intermediate domain, this seems sufficient for the calculation in practice of $F_0(r, E)$ at least for the values of E_0 quoted. Except for small E_0, the functions obtained show a roughly Gaussian behaviour only within a very small domain around the source. The range of practical interest is characterized by a transition from the Gaussian to the exponential dependence on r, a sort of quasi-exponential decay with a decreasing decay-length $\rightarrow \lambda_m$. The purely exponential decay predicted by the asymptotic formula (49) ought to begin to hold even at comparatively small distances. For $E_0 = 2$ Mev, for instance, a distance $r \gtrsim 25$ cm should be sufficient for this formula to be approximately valid for slow neutrons. Finally, a numerical comparison of our formulas with the age-approximation reveals that this approximation gives considerable errors even for $r = 0$ (values for $E_0 = 1$ of the magnitude $\approx 20\,\%$ too small, for example). But as this approximation predicts a slower decrease of $F_0(r, E)$ for small r than our formulas, it will also give values of F_0 which are too large within a small domain near the source, before it finally begins to decrease much faster than the actual exponential decay.

In conclusion, some comparisons with experiments will be given. In fact, several measurements of the density of slow neutrons in water have been made. Most of these concern thermal neutrons, but there are also many measurements of resonance neutrons, which are the main interest here. Unfortunately, the most used source, an $Ra - \alpha - Be$ source, has the energies of the primary neutrons almost continuously distributed within an energy range 1—10 Mev (roughly), so that an application of our formulas requires a superposition of terms corresponding to all the different values of E_0. This is rather laborious though possible in practice of course, if we know the primary spectrum of the $Ra - \alpha - Be$ source. But the knowledge of this seems to be incomplete. Recent measurements by TEUCHER [11] indicate two maxima, at about 4 and 6 Mev, respectively, in this spectrum. Qualitatively we may say then, that on account of the values of E_0 in question F_0 (or rather rF_0) must be expected to show an exponential behaviour already at distances $\approx$ 15—20 cm from the source (condi-

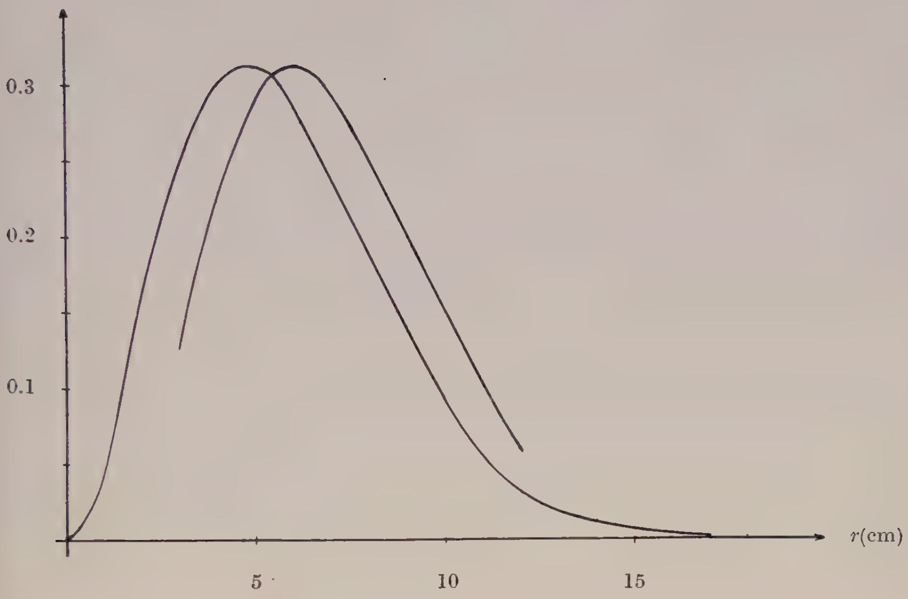

Fig. 1. $E_0 = 0.22$ Mev.

The curves drawn from $r = 0$ show the values of const. $r^2 F_0$, calculated for $E = 1.44$ ev.

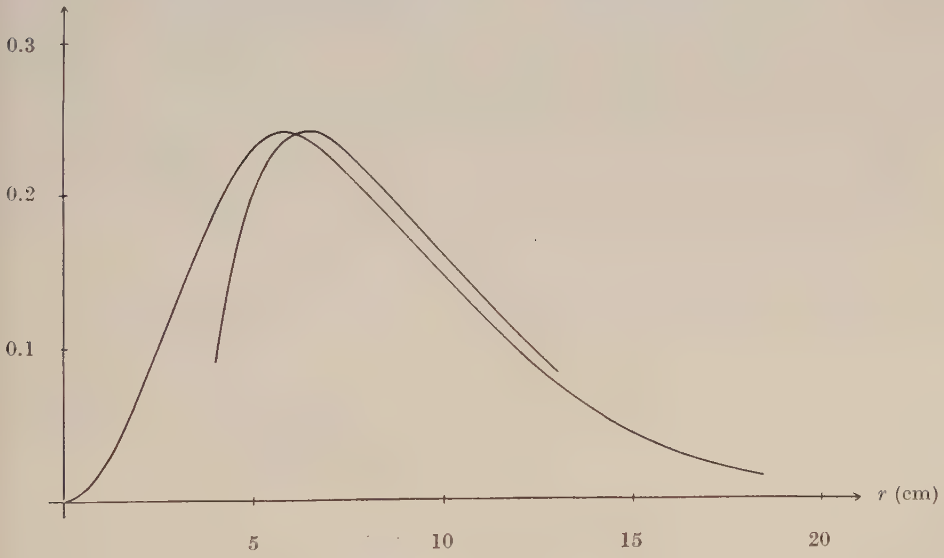

Fig. 2. $E_0 = 0.9$ Mev.

tion for (48) to be valid). The decay-length ought to be roughly $\approx$ the mean free path of the primary neutrons corresponding to the larger maximum $E_0 \approx 6$ Mev, or approximately ≈ 9 cm, which also is indicated by experiments, by RUSH [10], for example. Further the decay-length ought to be slowly increasing with r (in contrast to the case of a mono-energetic source), on account of the increasing importance for large r of the higher energy components in the primary spectrum.

Photo-neutron sources are in general better known and are often almost mono-energetic. For this case a comparison was made with measurements by O'NEAL [9]. In figures 1 and 2 the function $r^2 F_0$ is shown for D neutrons (= In resonance neutrons, $E = 1.44$ ev), calculated for $E_0 = 0.22$ Mev and 0.9 Mev respectively, using (37) and (48) (for $E_0 = 0.22$ also (44)) and assuming the same value of Q in both cases. These are the primary energies assumed for photo-neutron sources of RdTh + D_2O and RdTh + Be, respectively, and the measured neutron density from these sources, brought to the same maximum height as the theoretical curves, is also shown in the figures. As the source had an extension of roughly 4 cm, and r is the distance from its centre, a displacement of the measured curve from the theoretical curve to the right must be expected. This displacement ought to be near 2 cm in the neighbourhood of the source, and then decrease with increasing r. Taking this into account, the agreement between calculated and measured curves may be considered good.

Appendix

Calculation of the constant $\varrho = 2 - \alpha$. WICK [15] has shown that ϱ can be calculated as the lowest eigenvalue of a certain differential equation ([15] eq. (105)), formally corresponding to the radial Schrödinger equation for a two dimensional particle. For the application to water this equation must first be generalized to a medium containing more than one slowing-down element, which is possible when the mean free path $\lambda_i(E)$ in the component i varies with E in the same way as the total mean free path $\lambda(E)$. Then we take $\lambda / \lambda_i = \varkappa_i = $ constant ($i = 1, 2, \ldots, \sum_i \varkappa_i = 1$). If we assume for simplicity only two different nuclei ($\varkappa_1 + \varkappa_2 = 1$), with masses M_1 and M_2 neutron masses, respectively, the equation is

$$\left\{\frac{d^2}{d\sigma^2} + \frac{1}{\sigma}\frac{d}{d\sigma} + \frac{4\gamma}{M_1}(2\varrho - 3) + A_1 e^{-\sigma^2} + A_2 e^{-\sigma^2 \frac{M_2}{M_1}} - \frac{4\gamma^2}{M_1^2}\sigma^2\right\} U(\sigma) = 0, \tag{62}$$

with

$$A_1 = 2\varkappa_1 \frac{(M_1 + 1)^2}{M_1^2}, \quad A_2 = 2\varkappa_2 \frac{(M_2 + 1)^2}{M_1 M_2}. \tag{63}$$

Here we must always have $\gamma > 0$, given then by

$$\gamma = E_0 \lambda'(E_0) / \lambda(E_0) = \sum_\nu d_\nu \lambda_\nu / \lambda_m, \tag{64}$$

if (1) is used. For the λ variation actually treated numerically, (1′) we obtain

$$\gamma = d\lambda_1/\lambda_m. \tag{64′}$$

At first sight (62) looks suspect, but it is in fact symmetrical in M_1 and M_2, which is easily seen by the transformation $\xi = \sigma\sqrt{\frac{M_2}{M_1}}$. The generalisation to more than two components is obvious.

The lowest eigenvalue ϱ of (62) can be calculated by the variation method. WICK [15] has used the eigenfunction $\exp(-s\sigma^2/2)$ with s as the variable parameter for a rough computation for carbon. Using the general equation (62), the minimum condition is

$$\frac{4\gamma^2}{M_1^2} + A_1 a_1^2 + \frac{M_2}{M_1} A_2 a_2^2 = s^2, \quad a_1 = \frac{s}{1+s}, \quad a_2 = \frac{s}{M_2/M_1 + s}. \tag{65}$$

However, this method was found to give too bad values of ϱ. But (65) was still used for calculation of a rough value s_0 for s. In order to obtain a better value of ϱ we then tried a linear variation function

$$U(\sigma) = \sum_{n=0} c_n z^{2n} e^{-z^2}, \quad \text{with} \quad z = \sigma\sqrt{s_0/2}, \tag{66}$$

making use in this way of the approximate knowledge of the eigenfunction, which results from (65). In (66) s_0 is a constant, taken roughly as the root of (65), c_n are independent variables, which have to be calculated from the usual system of linear, homogeneous equations. This also contains the eigenvalue as a parameter, which is the root of a secular equation

$$\text{Det.}\,|H_{nn'}| = 0, \qquad (n, n' = 0, 1, \ldots), \tag{67}$$

where the elements of the determinant have the form

$$H_{nn'} = \frac{(n+n')!}{2^{n+n'+2}} \Bigg[A_1 a_1^{n+n'+1} + A_2 a_2^{n+n'+1} - \Phi + + s_0 \frac{(n-n')^2 - (n+n')}{n+n'} - \frac{4\gamma^2}{s_0 M_1^2}(n+n'+1) \Bigg]. \tag{68}$$

Here a_1 and a_2 have the form given in (65), inserting s_0 for s. Φ is the parameter to be determined by solution of (67), related to the lowest eigenvalue ϱ of (62) by the equation

$$\varrho \leq 1.5 - \frac{M_1}{8\gamma} \Phi_m. \tag{69}$$

Then Φ_m is the largest root of (67). This must be determined numerically by inserting trial values of Φ in (67). These values can be estimated with the help of the roots of (67) for the determinants of the first and the second order, which can be computed directly. In practice it was only necessary to take two or three trial values of Φ, and then calculate the root Φ_m by interpolation. However, using a determinant of the kth order in (67), the value thus obtained is still only an approximation $\Phi_m^{(k)}$ of Φ_m, always smaller than the correct value. Therefore, the successive approximations $\Phi_m^{(1)}$, $\Phi_m^{(2)}$, . . . were also calculated and the convergence studied. Then it seemed sufficient to use the sixth order determinant in (67), and the last figure in the value chosen for Φ_m was obtained by estimation from the successive approximations. For the numerical evaluation of (67) for each trial value of Φ, the method devised by JAMES and COOLIDGE [7] was used. In this way we also made an estimation of the second eigenvalue ϱ_1.

For the application to carbon the formulas are simplified because we have $\varkappa_1 = 1$, $\varkappa_2 = 0$. As an example, for $E_0 = 2$ Mev, we took $s_0 = 0.7$ and the successive approximations of Φ_m are: 0.251, 0.260, 0.308, 0.309, 0.3149, 0.3151, and the value 0.3160 was chosen for Φ_m. For water we took hydrogen as nucleus 1, thus $M_1 = 1$, $\varkappa_1 = 0.87$, and $M_2 = 16$, $\varkappa_2 = 0.13$. The terms corresponding to oxygen then only appear as small corrections in (62), (65) and (68). Taking then $E_0 = 2$ Mev, for example, we used $s_0 = 2.6$, obtaining for Φ_m the following successive approximations: 2.47, 2.47, 2.52, 2.52, 2.525, 2.527, and for Φ_m the value 2.531 was chosen.

SUMMARY

The stationary space energy distribution of slow neutrons in an infinite, homogeneous slowing-down medium, where the mean free path of the neutrons $\lambda(E)$ depends on the energy E, has been studied. $\lambda(E)$ is assumed to have the form $\lambda(E) = \sum_\nu \lambda_\nu (E/E_0)^{d_\nu}$ $(d_\nu \geq 0,\ d_0 = 0)$, where E_0 is the energy of the neutron source. This is supposed to be an isotropic point source. Firstly an analytic expression for the collision density $F_0(r, E)$ has been calculated (r = distance from the source) which is valid for a slowly varying $\lambda(E)$ and at sufficiently small distances and contains the first corrections to age-theory. Secondly, the case of practical importance when $\lambda(E)$ decreases with decreasing E has been treated and general expressions for $F_0(r, E)$, covering essentially all values of r, have been calculated. The term which dominates the behaviour of $F_0(r, E)$ undergoes a transition from a roughly Gaussian to an exponential dependence on r. In particular, for $r \to \infty$ the function calculated shows the dependence on r and E predicted by WICK. The numerical constant in the asymptotic formula is also obtained, as well as detailed conditions for the validity of this formula. Numerical applications have been made to carbon and water. Comparisons show good agreement with experiments for water.

The problem of the slowing-down of neutrons was suggested to me by Professor IVAR WALLER, and I wish to express my gratitude to him for his encouraging interest in my work and for valuable discussions and suggestions.

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